Solutions for Section 4.4

1. Compute the length of the graph of $y = x^2$ for x in $[-1, 1]$.

   Parametrize the graph by $\vec{r}(t) = \langle t, t^2 \rangle$ with t in $[-1, 1]$. Compute $\vec{r}\,'(t) = \langle 1, 2t \rangle$ to get $\|\vec{r}\,'(t)\| = \sqrt{1 + 4t^2}$. The length of the curve is then

   $$L = \int_{-1}^{1} \sqrt{1 + 4t^2}\, dt \approx 2.96$$

   where the definite integral was approximated using the fnInt feature of a TI-86 calculator.

2. Compute the length of the graph of the sine function for one complete period.

   Parametrize the graph by $\vec{r}(t) = \langle t, \sin t \rangle$ with t in $[0, 2\pi]$. Compute $\vec{r}\,'(t) = \langle 1, \cos t \rangle$ to get $\|\vec{r}\,'(t)\| = \sqrt{1 + \cos^2 t}$. The length of the curve is then

   $$L = \int_{0}^{2\pi} \sqrt{1 + \cos^2 t}\, dt \approx 7.64$$

   where the definite integral was approximated using the fnInt feature of a TI-86 calculator.

3. Compute the length of the helix $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ for t in $[0, 2\pi]$.

   Compute $\vec{r}\,'(t) = \langle -\sin t, \cos t, t \rangle$ to get $\|\vec{r}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$. The length of the curve is then

   $$L = \int_{0}^{2\pi} \sqrt{2}\, dt = \sqrt{2}(2\pi) = 2\pi\sqrt{2}.$$

4. For a curve parametrized by $\vec{r}(t)$ for t in $[a, b]$, the *arclength function* is defined by

   $$s(t) = \int_{a}^{t} \|\vec{r}\,'(u)\|\, du.$$

   (a) Explain what the output $s(t)$ means for a given input t.

   The output $s(t)$ gives the length of the piece of the curve from the initial point $\vec{r}(a)$ to the point $\vec{r}(t)$.

   (b) What is the meaning of $s(b)$?

   The output $s(b)$ gives the total length of the curve from the initial point $\vec{r}(a)$ to the final point $\vec{r}(b)$.

   (c) Find $s'(t)$.

   By the First Fundamental Theorem of Calculus, $s'(t) = \|\vec{r}\,'(t)\|$. Note that the input t must be used on both sides of the equality to make sense.

   (d) Explain why this result for $s'(t)$ makes sense.

   The derivative $s'(t)$ can be interpreted as the rate of change in the length of the curve that is traced out in time t. The quantity $\|\vec{r}\,'(t)\|$ is the speed at which the curve is traced out. It makes sense that the rate of change in the length of the curve that is traced out is equal to the speed at which the curve is traced out.