Computing directional derivatives

Suppose we have a function $f : \mathbb{R}^2 \to \mathbb{R}$ and a point P in the domain of f. For the infinitesimal displacement $d\vec{r}$, the corresponding change in f is given by

$$df = \vec{\nabla} f \cdot d\vec{r} \tag{1}$$

which we can think of as a rise. The corresponding run is given by the magnitude of $d\vec{r}$. For notational convenience, we will denote this magnitude as $ds = \|d\vec{r}\|$.

1. Draw a picture showing
   - a point P,
   - the level curve (labeled f_0) passing through P,
   - a displacement $d\vec{r}$ at P that is *not* perpendicular to the level curve through P,
   - the level curve labeled $f_0 + df$, and
   - the gradient vector $\vec{\nabla} f$ at P.

2. Divide both sides of Equation (1) by ds.

3. What does the left side of the equation in Step 2 represent?

4. On the right side of the equation you get from Step 2, you should be able group some of the factors to form a unit vector. Name this unit vector $\hat{u}$. Rewrite the equation using $\hat{u}$. Include $\hat{u}$ in your picture from Step 1.

5. Complete the following sentence that describes one way to read the equation in Step 4:

   To compute the rate of change in f in the direction $\hat{u}$,

6. For the point $(1, 2)$, compute the rate of change in $f(x, y) = x^2y^3$ in the direction $\hat{u} = \frac{4}{5}\,\hat{\imath} + \frac{3}{5}\,\hat{\jmath}$. Include a rough sketch of the relevant point and vectors.

Answer: $\dfrac{df}{ds} = 20$

7. For the point $(-2, 4)$, compute the rate of change in $f(x, y) = x\sqrt{y}$ in the direction of the vector $\vec{v} = \hat{\imath} - 3\,\hat{\jmath}$. Include a rough sketch of the relevant point and vectors.

Answer: $\dfrac{df}{ds} = \dfrac{7}{2\sqrt{10}}$

8. For the point $(1, 2)$, compute the rate of change in $f(x, y) = y\sin(\pi x)$ in the direction of the point $(3, 7)$. Include a rough sketch of the relevant points and vectors.

Answer: $\dfrac{df}{ds} = \dfrac{-4\pi}{\sqrt{29}}$