Free Groups

Recall

Knowing that all elements of a group are generated by a few generators and also knowing the "pertinent" relations amongst those generators allows for "easy" computation in the group.
$D_3 = \langle r, f; r^3 = e, f^2 = e, rf = fr^{-1} \rangle = \{e, r, r^2, f, rf, r^2 f\}$

What about $\langle r, f; \ \rangle$? That is, what if there are **no** relations?

Historical Description of Free Groups

Definitions and Notation

Basis We use a, finite or infinite, set $S = \{a, b, \cdots\}$ as the **basis** for our free group. This set is sometimes called the *alphabet* of the free group and the elements are called **generators**.

Words on S A word on the basis set is any finite string of symbols from the set. Repetition is allowed. For example, $abbbca$ is a word on the set $\{a, b, c, d\}$. The set of all words on the set S is denoted $W(S)$. Notate the **empty word** by 1.

Composition Define an operation, law of composition, $\cdot$

$$\cdot : W(S) \times W(S) \longrightarrow W(S)$$

by juxtaposition. For example, $(aa) \cdot (ba) = aaba$ and $(1) \cdot (abc) = abc$.

Note that $W(S)$ with this operation is a semigroup:

1. $W(S)$ is closed under $\cdot$.
2. Associativity holds.
3. There is an identity element.

Extended Words Define $S' = \{a', b', \cdots\}$ and $W' = W(S \cup S')$, the set of all words on the set $S \cup S'$.

Cancelation If a word in W' contains an occurence of either xx' or $x'x$ for some $x \in S$ we say we **cancel** x in this word by removing the pair, giving a shorter word (possibly 1).

Given any word $w \in W'$, after a finite sequence of cancelations, we obtain a word w_0 with no possible cancelations remaining. We call w_0 a **reduced word** and say it is a reduced form of w.

Example The word $w = babb^{-1}a^{-1}c^{-1}ca$ reduces to ba but the reduction process can be carried out in more than one way.

Proposition 1 *The reduced form of a word in W' is unique.*

Hint: Induction and define $(x')' = x$ for any $x \in W$.

Definition 1 *Two words $w_1, w_2 \in W'$ are **related**, $w_1 \sim w_2$ if they have the same reduced form.*

Proposition 2 *The relation $\sim$ is an equivalence relation.*

Proof is easy.

Proposition 3 *If $w_1 \sim w_2$ and $v_1 \sim v_2$, then $w_1 v_1 \sim w_2 v_2$.*

Intuitive but takes time to express clearly.

Definition 2 *Let $F[S]$ denote the set of equivalence classes of words in W' and define an operation, $*$ on $F(S)$ by $[w_1] * [w_2] = [w_1 \cdot w_2]$.*

Proposition 4
1. *The operation $*$ is well-defined*
2. *$F[S]$ with this operation is a group — called the **Free Group on S.***

More Notation

1. When working in the free group, we will suppress the "[]" notation.
2. For any positive integer n, we abbreviate n adjacent occurrences of an element of S, say $aa \cdots a$ by a^n and n occurrences of a^{-1} by a^{-n}.

Example The infinite cyclic group $\langle a \rangle$ is a free group on $S = \{a\}$. But free groups on two generators are much more complicated.
Such groups are called free groups on their generating sets because they consist precisely of the elements obtainable from the generators by freely using the defining properties of a group.

Theorem 5 *If $F[S], F[T]$ are free groups, then $F[S] \cong F[T]$ if and only if $|S| = |T|$. This is true even if S, T are infinite.*

Proof. One direction is trivial if the generating sets are finite. It is also trivial for those familiar with cardinality of infinite sets. The other direction is taken care of by Exercise 6 of Free Groups at the end of these notes. ■

Functional Characterization

Theorem 6 *Suppose F is a free group and $A \subset F$.*
If for each group G, and for each set-function $\phi : A \longrightarrow G$, there is unique $\phi^* : F \longrightarrow G$ satisfying

1. *ϕ^* is a homomorphism.*
2. *$\phi^*(a) = \phi(a)$ for each $a \in A$,*

then, F is a free group on the set A.

Modern Approach to Free Groups

A more modern approach to free groups is to use Theorem 6 as the definition of a free group on the set A and then show that the groups consisting of the equivalence classes of words satisfy this definition.
The modern approach is based on properties of FUNCTIONS which makes it easier to move up several levels of abstraction.
DIAGRAM OF UNIVERSAL PROPERTY
Reprove $\Longleftarrow$ direction of Theorem 5 using the modern approach.

Generators and Relations

We say that a set A contained in a group G **generates** G if G is the homomorphic image of the free group on A, $F[A]$.
DRAW DIAGRAM HERE

- First Isomorphism Theorem tells us that $G \cong F[A]/\ker(f^*)$
- The elements in $N = \ker(f^*)$ are called **relations** among the generators. Note that if w is a relation, then $f^*(w) = 1_G$.
- If f^* is an isomorphism, then we say G is a free group too.

Computations

Recall how easy it was to make computations in the group D_3.
Note, given $G \cong F[A]/N$, if we know the generators of $F[A]$ and all the relations N, then we should be able to "easily" compute in G.
But even in $D_3 = \langle r, f; r^3 = e, f^2 = e, rf = fr^{-1}\rangle$ the number of relations is infinite. What helped us there is that we had three relations that we used to simplify all computations. This worked because those three words are generators of the normal subgroup $N = \ker(f^*)$.
Generalizing, if $G \cong F[A]/N$, then a set of words $R = \{r_1, \cdots, r_t\}$ is called a set of **defining relations** for G if

1. $R \subset G$
2. N is the "smallest" normal subgroup containing the set R.

Here, "smallest" means N is a subgroup of every normal subgroup of G that contains the set R. Exercise 5 on Generators and Relations addresses this.

So now we KNOW a different way of thinking about

$$D_n = \langle r, f; r^3, f^2, rfrf\rangle$$

and why we write it as

$$D_n = \langle r, f; r^n = e, f^2 = e, rf = fr^{(n-1)}\rangle$$

Exercises on Free Groups

1. Prove that the reduced form of a word in W' is unique.
2. Prove or disprove: The free group on two generators is isomorphic to the product of two infinite cyclic groups.
3. (a) Let F be the free group on x, y. Prove that the two elements $u = x^2$ and $v = y^3$ generate a subgroup of F which is isomorphic to the free group on u, v.
   (b) Prove that the three elements $u = x^2, v = y^2$ and $z = xy$ generate a subgroup isomorphic to the free group on u, v, z.
4. Define a **closed word** in S' to be the oriented loop obtained by joining the ends of a word. Thus, a closed word of the form (abcd) would be the same closed word as (bcda) or (dabc). Establish a bijection between the set of all reduced closed words and the set of conjugacy classes in the free group.
5. Deleted
6. Let $F[A]$ be the free group on a finite set A and let N be the set of all products of squares of elements in the free group: $F[A] = \{w_1^2 w_2^2 \cdots w_n^2 : w_i \in F[A], n \in \mathbb{Z}^+\}$
   (a) prove that N is a normal subgroup of $F[A]$.
   (b) Prove that $F[A]/N$ is an abelian 2-group
   (c) Prove that the size of $F[A]/N$ is $2^{|A|}$.
   (d) You need not prove it but this is also true if A is an infinite set .
   (e) Deduce that if two free groups are isomorphic, then the cardinalities of their generating sets are equal. This verifies the forward direction of Theorem 5.
7. Prove that every group is the homomorphic image of a free group. [Hint: Theorem 6]

Exercises on Generators and Relations

1. Prove that two elements a, b of a group generate the same subgroup as bab^2, bab^3.
2. Deleted
3. Prove or disprove: In a group G, y^2x^2 is in the normal subgroup generated by xy and its conjugates.
4. Prove that the group generated by x, y, z with the single relation $yxyz^{-2} = 1$ is actually a free group.
5. If $G \cong F[A]/N$
   (a) prove that $N = \cap\{K : K \text{is a normal subgroup of } G \text{ containing } R\}$
   (b) prove that $N =$ the set of all finite products of conjugates of elements of R