Fields Setup

Given $\alpha \in K \supseteq F$ (K is an extension field of F)

If α is either transcendental or algebraic over F

- $F[x]$ is the **ring** of polynomials in the variable x with coefficients in F.
- $F(x)$ is the field of fractions built from the integral domain $F[x]$ of all polynomials in x with coefficients in F.

    1. $F(x)$ is the quotient field of $F[x]$.

- $F[\alpha]$ is the **Ring** generated by $\alpha \in K$ over F

    1. $F[\alpha] = \{a_n\alpha^n + \cdots + a_0 \in K : a_i \in F\}$
    2. Special case: when we adjoin an element α to F
        (a) Note that adjunction requires α to be algebraic over F and we know
        $$\frac{F[x]}{\langle f(x)\rangle} = F[\alpha]$$

- $F(\alpha)$ is the smallest **subfield** of K containing both F and α.

    1. Called the field extension of F generated by the element $\alpha \in K$.
    2. $F(\alpha) = \{a + b_1\alpha + \cdots + b_n\alpha^n : a, b_i \in F\}$
    3. This extends naturally to $\alpha_1, \cdots, \alpha_n$

- $F(\alpha)$ is isomorphic to the field of fractions of the **ring** $F[\alpha]$ (no notation has been defined for this)

    1. This is true because of closure in one direction and minimality in the other.

If α is transcendental

Proposition 1 *If α is transcendental over F then*

1. *The map $\psi : F[x] \to F[\alpha]$ given by $\psi(f(x)) = f(\alpha)$ is an isomorphism.*
2. *Hence $F(\alpha)$ is isomorphic to $F(x)$,the fraction field of $F[x]$, of rational functions over the field F.*

Remark 1 *Note that for any two transcendental elements α , β we have $F(\alpha) \approx F(\beta)$.*
Thus, $\mathbf{Q}(\pi) \approx \mathbf{Q}(e)$ where the isomorphism takes π to e.
This is surprising at first glance but careful consideration shows the isomorphism cannot be continuous when the fields are regarded as subfields of the real numbers. (Think of rational numbers close to π).

If α is algebraic

Proposition 2 *Suppose α is algebraic over F and $f(x)$ is the irreducible polynomial for α over F, then*

1. *.The map*
$$\psi : \frac{F[x]}{\langle f(x) \rangle} \to F[\alpha]$$
is an isomorphism

2. *$F[\alpha]$ is a field and so $F[\alpha] = F(\alpha)$ by closure and minimality*

3. *More generally, if $\alpha_1, \cdots, \alpha_n$ are algebraic elements of a field extension K of F then*
$$F[\alpha_1, \cdots, \alpha_n] = F(\alpha_1, \cdots, \alpha_n)$$

Proposition 3 *Let α be an algebraic element over a field F and let $f(x)$ be its irreducible polynomial. Suppose f has degree n. Then $(1, \alpha, \cdots, \alpha^{n-1})$ is a basis for $F[\alpha]$ as a vector space with scalars in the field F.*
Proof: *We have already proved this in the more general setting of rings.*

Corollary 4 *If α, β are algebraic elements that generate isomorphic extension fields over F, then their irreducible polynomials must have the same degree.*

Remark 2 *This is a necessary but not a sufficient condition for extension fields to be isomorphic.*

- *In Class Problem #7?: Prove or disprove:* $\mathbf{Q}\left(\sqrt{-1}\right)$ *is isomorphic to* $\mathbf{Q}\left(\sqrt{-5}\right)$.

Interesting Case: (setting up Galois Theory)

We want to investigate where there is an isomorphism from $F(\alpha)$ to $F(\beta)$ that fixes every element of F and maps α to β.

Proposition 5 *(Equal Irreducible Polynomials) Let $\alpha \in K$ and $\beta \in L$ be algebraic elements of two extension fields of F.There is an isomorphism $\sigma : F(\alpha) \to F(\beta)$ which is the identity on the subfield F and which sends α to β if and only if the irreducible polynomials for α and β over F are equal.*

Definition 1 *Let K and K' be two extensions of the same field F. An isomorphism $\phi : K \to K'$ which is the identity on elements of F is called an* isomorphism of field extensions, or an F -isomorphism. *Two extensions K, K' of F are said to be* isomorphic field extensions *if there exists an F isomorphism taking K to K'.*

Proposition 6 *(In Class Homework #8) Let $\phi : K \to K'$ be an isomorphism of field extensions of F and let $f(x)$ be a polynomial with coefficients in F. Let α be a root of f in K and let $\alpha' = \phi(\alpha)$ be its image in K'.Then*

1. *α' is also a root of f.*

2. *Deduce the irreducible polynomials for K and K' are equal.*