Powers of Diagonal Matrices

Definition: A matrix $D \in M_{mm}$ is diagonal if for all i, j satisfying $1 \le i, j \le m$

$$[D]_{i,j} = 0 \text{ when } i \neq j$$

Lemma IC-1: If A and B are $m \times m$ diagonal matrices, then AB is also diagonal and $[AB]_{ij} = [A]_{ij}[B]_{ij}$ for all i, j satisfying $1 \le i, j \le m$.

Proof: Using matrix multiplication we know that for all i and j, $[AB]_{ij} = \sum_{l=1}^{m} [A]_{il}[B]_{lj}$. Using the fact that A and B are diagonal we also know that $[A]_{il} = 0$ when $i \neq l$ and $[B]_{lj} = 0$ when $l \neq j$.Thus,

$$[A]_{il}[B]_{lj} = \begin{cases} 0 \text{ if } i \neq l \\ 0 \text{ if } l \neq j \\ [A]_{ii}[B]_{ii} \text{ if } i = l = j \end{cases}$$

If $i \neq j$ then l can never equal both i and j, so $[AB]_{ij} = \sum_{l=1}^{m} [A]_{il}[B]_{lj} = \sum_{l=1}^{m} 0 = 0$ which tells us that AB is a diagonal matrix. In addition. if $i = j$ then the only nonzero term in the sum occurs when $l = i = j$ and $[AB]_{ii} = \sum_{l=1}^{m} [A]_{il}[B]_{lj} = [A]_{ii}[B]_{ii}$ .Thus, in all cases, $[AB]_{ij} = [A]_{ij}[B]_{ij}$ and AB is diagonal.

Lemma IC-2: If D is an $m \times m$ diagonal matrix, then for each positive integer n, D^n is a diagonal matrix and $[D^n]_{ij} = [D]_{ij}^n$ for all i, j satisfying $1 \le i, j \le m$.

Proof: We use mathematical induction in our proof.

1. Base case: If $n = 1$ then $D^n = D^1 = D$ is diagonal by hypothesis and $[D^1]_{ij} = [D]_{ij}^1$ for all i, j satisfying $1 \le i, j \le m$.

2. For our inductive hypothesis we assume that k is a positive integer for which D^k is diagonal and $\left[D^k\right]_{ij} = [D]_{ij}^k$ and we consider the matrix D^{k+1}. Since both D^k and D are diagonal, then Lemma IC-1 implies that $D^{k+1} = D^k D$ is diagonal. Thus

$$\begin{aligned} \left[D^{k+1}\right]_{ij} &= \left[D^k\right]_{ij}[D]_{ij} && \text{by Lemma IC-1} \\ &= [D]_{ij}^k [D]_{ij} && \text{by the Inductive Hypothesis} \\ &= [D]_{ij}^{k+1} && \text{by multiplication of Complex numbers.} \end{aligned}$$

The principle of mathematical induction now implies that if D is a diagonal matrix, then for every positive integer n, D^n is also diagonal and $[D^n]_{ij} = [D]_{ij}^n$ for all i, j satisfying $1 \le i, j \le m$.