The Basics of Set Theory

Sets and Set Membership Think of sets as abstract collections of objects. The objects that are collected in a set are called its *members* or *elements*. So the letter α is an element of the set consisting of the letters b, α, t.

Notation

For us,

1. Capital letters: $A, B, C, \cdots$ represent sets
2. Lower case letters represent members of sets: $a, b, c, \cdots$

The principle concept of Set Theory is *belonging* or *membership*. That is, elements are members (or belong to) sets. The notation we use to denote this is the symbol $\in$ and we denote its logical negation as $\notin$.

Note that sets themselves can be the elements of sets. For example, a state is a collection of counties and our country is a set of states.

Calculus is the study of properties of certain sets (and sets of sets) of real numbers.

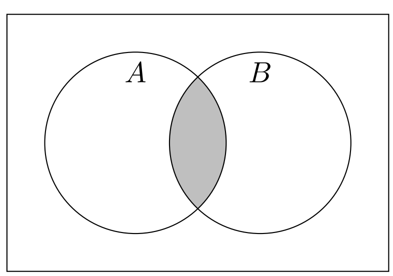
Specifying a set

List Notation The convention is to use curly braces to denote sets and, if the set is easily described, to list the elements, separated by commas. The order in which we list them is not important — nor, since belonging is what is important, is how many times we list a particular element. For example, all three of the following represent the same five-element set.

$$\{1, 2, 3, 4, \{2, 3\}\} = \{1, 3, \{2, 3\}, 2, 4\} = \{1, 1, 1, 2, 3, 3, 4, \{2, 3\}\}.$$

Also note that sets themselves can be elements of other sets. For example, $\{2, 3\} \in \{1, 3, \{2, 3\}, 2, 4\}$.

Venn Diagrams Sets can sometimes be represented visually by using *Venn Diagrams*. For example, we can represent the set of all elements that are simultaneously in sets A and B with the diagram



Predicate Notation Another method of specifying a set is *Predicate Notation* where we specify a condition that all of the elements in a set meet. For example $\{x \mid x \text{ is a natural number}\}$ which is read as "the set of all x such that x is a natural number" is a different way of specifying the set $\{1, 2, 3, \cdots\}$.

The curly braces tell us this is a set, the letter x is a variable that stands in for any object meeting the conditions described after the vertical line. The vertical line is spoken "such that".

It is possible to describe a set in many different ways but the manner in which it is defined is irrelevant; its' membership is what is important.

It turns out that predicate notation can describe so many things that it is too powerful and can generate paradoxes. Here is an example of the difficulty. It is due to Bertrand Russell and is named for him.

Let $X = \{y \mid y \notin y\}$ Could X be a member of itself?

Suppose $X \in X$. Then $X \in \{y \mid y \notin y\}$ which implies, substituting X in for y, that $X \notin X$, which contradicts our supposition.

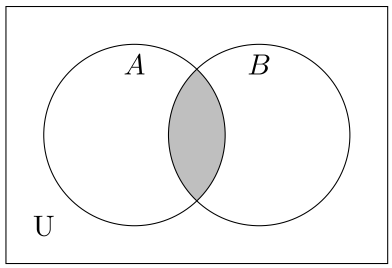
Suppose $X \notin X$. Then the predicate notation tells us that X is an allowable item to substitute in for y which implies $X \in X$ which also contradicts our supposition — paradox!

To avoid this and other paradoxes, mathematicians restrict the predicate notation by always specifying the elements of the set being defined must come from a previously specified set (which is called the "universe". Here is an example where $U = \{1, 2, 3, 4, 5, 6\}$.

$$B = \{x \in U \mid x \text{ is an even integer}\}$$

Note that this set B can also be listed as $B = \{2, 4, 6\}$.

In Venn diagrams above, the universe is depicted by the box surrounding the circles.



Set-Theoretic Relations

Equality Two sets are equal if and only if they have the same elements. If set A equals set B, we write $A = B$.

Subsets Set A is a *subset* of set B if and only if every element of A is an element of B. If A is a subset of B, we write $A \subseteq B$. In particular, $\{A \subseteq A\}$. **Union** The *union* of set A with set B is the set that contains all the elements that belong to A or B (note that this is the logical "or" discussed in the logic handout) and is written $A \cup B$. In symbols this says

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B.\}$$

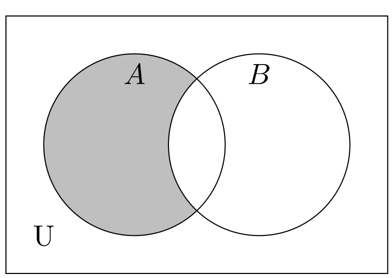
Intersection The *intersection* of set A with set B is the set that contains all the elements that belong to both A and B (note that this is the logical "and" discussed in the logic handout). It is written $A \cap B$, and is depicted in the second Venn diagram above. In symbols this says

$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B.\}$$

Set Difference The *difference* between a set A and a set B is the set of all elements that belong to A but not to B. It is written $A \setminus B$, is defined by

$$A \setminus B = \{x \in U \mid x \in A \text{ and } x \notin B\},$$

and is depicted by



Empty Set The *empty set* is the set that has no elements. It is written either $\{\}$ or $\emptyset$ and is defined by

$$\emptyset = \{x \in U \mid x \neq x\}.$$

Recall the definition of subset: $A \subseteq B$ if and only if **every** element of A is an element of B. This is a universally quantified statement and can be written

$$(A \subseteq B) \Leftrightarrow (\forall x \in A, \quad x \in B).$$

Now consider whether or not the claim "the empty set is a subset of every set B" is true or false. This can be rewritten as "**every** element of $\emptyset$ is an element of B". This is a universally quantified statement that is either true or false. But, if it were false then it's negation would be true. Our logic handout tells us that this negation is "there is an element of $\emptyset$ that is not in B". But this can't be true because $\emptyset$ has no elements at all. Thus, the statement "**every** element of $\emptyset$ is an element of B" is true which tells us that the empty set is a subset of every set.

Ordered Pairs

An *ordered pair* is a sequence of two elements in which the order they are written is important. If the elements are α and β, then we would write the ordered pair (where α is first) using parentheses: (α, β). Note that this is a different ordered pair from (β, α). Note also that the same element can be in both positions as in (α, α).

Cartesian Product Suppose A and B are sets. Form all possible ordered pairs that have an element of A in first position and an element of B in second position. The set of all of these ordered pairs is called the **Cartesian Product of A and B**, is written $A \times B$, and is defined in symbols by

$$A \times B = \{(\alpha, \beta) \mid \alpha \in A \text{ and } \beta \in B\}.$$

Relations and Functions

Relations A subset of $A \times B$ is called a relation between set A and set B. The set A is called the *domain* of the relation and the set B is called the *codomain* of the relation. Relations are important in computer science (e.g., relational databases) and logic but for calculus, we only consider a very special type of relation that is called a **function**.

A function is a relation with the following restriction: there cannot be two ordered pairs in the function that have the same first coordinate but different second coordinates. We typically denote functions using the letters f, g, h, F, G, H etc. and use notation to specify the domain and codomain. For example, in your earlier math classes you used the squaring function. If we name this function f, then a careful notation for specifying it, using $\mathbb{R}$ to denote the set of all real numbers, is:

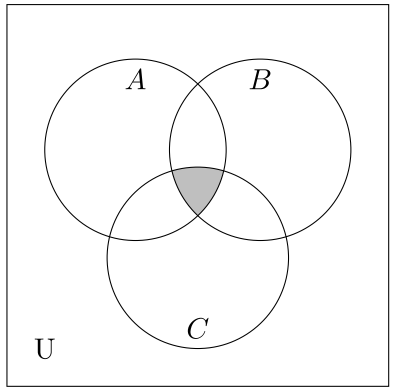
$$f = \{(x, x^2) \mid x \in \mathbb{R}.\}$$

Because scientists think of functions as taking inputs from the domain set and acting on them to produce outputs from the codomain set, we will also use the following method (illustrated on the squaring function) to describe a function:

$$f : \mathbb{R} \longrightarrow \mathbb{R} \qquad (1)$$

$$x \mapsto f(x) = x^2 \qquad (2)$$

Here is a Venn diagram for $A \cap B \cap C$.



Problems

1. Draw the Venn diagrams for
   (a) Draw the Venn diagram for $A \cup (B \cap C)$.
   (b) Draw the Venn diagram for $A \cap (B \cup C)$.
   (c) Draw the Venn diagram for $(A \cap B) \cup (A \cap C)$.
   (d) Draw the Venn diagram for $U \setminus (A \cup B)$.
   (e) Draw the Venn diagram for $(U \setminus A) \cap (U \setminus B)$.
2. Write out the Cartesian product $A \times B$ where $A = \{roy, ann, teri\}$ and $B = \{2, 7\}$.
3. Using the definitions but in your own words, explain why $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. [Hint: see the logic handout.]