# Symmetric Lie Groups and Conservation Laws in Physics

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## Abstract

This paper examines how conservation laws in physics can be found from analyzing the symmetric Lie groups of certain physical systems. To begin, I will give some background on the physics of Lagrangian mechanics. Then I will discuss Lie symmetries, Lie groups, and Lie algebras. This will include an exposition on continuous groups, tangent vectors, and infinitesimal generators. I will end with an example using the infinitesimal generators of a Lie algebra to derive conservation of energy for the motion of a free particle in special relativity.

## 1 Lagrangian Mechanics

A common objective in theoretical physics is to examine a given physical system and predict how the system changes qualitatively and quantitatively as time increases. One way of doing this, which is discussed in any introductory physics course, is to find all the forces acting on the system and use Newton's Third Law, $\overrightarrow{F} = m\overrightarrow{a} = \dot{\overrightarrow{p}}$, to predict future behavior. This is called *Newtonian mechanics.*

In more advanced mechanics courses, another method is introduced, called *Lagrangian mechanics*; instead of looking at forces, we look at energy. For more complicated physical situations, working with the energies of a system is almost always easier than working with forces. However, this method is not introduced in lower division courses as it requires a familiarity with calculus of variations.

The *Lagrangian* $\mathcal{L}$ is given by

$$\mathcal{L} = T - V \tag{1}$$

where $T$ is kinetic energy and $V$ is potential energy. The *action* $S$ is defined as

$$S = \int_{x_1}^{x_2} \mathcal{L}[x, \dot{x}, t]\, dt \tag{2}$$

and the equation of motion for the system is given by $x(t)$ such that $S$ is stationary (an extremum). This leads to the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x} \quad \text{for} \quad \mathcal{L}[x, \dot{x}, t] \tag{3}$$

**Note:** The Lagrangian may be a function of several variables, in which case the Euler-Lagrange equation becomes a set of equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i} \quad \text{for} \quad \mathcal{L}[q_1, q_2, \ldots, q_i, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_i, t] \tag{4}$$

**Example 1.1** Consider a mass on a spring, as seen in Figure 1. When the mass is displaced from its rest position, the spring exerts a force back towards equilibrium. The potential energy associated with this is $V = \frac{1}{2}kx^2$, where $k$ is the spring constant and $x$ is the displacement of the mass from equilibrium. The kinetic energy associated with the velocity of the mass is $T = \frac{1}{2}m\dot{x}^2$, where $m$ is mass and $\dot{x}$ is the first time derivative of the mass's displacement, i.e. its velocity. Thus the Lagrangian for this system is

$$\mathcal{L}[x, \dot{x}, t] = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{5}$$

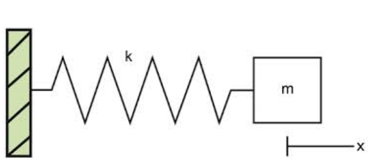


Figure 1: A mass $m$ on a spring with spring constant $k$.

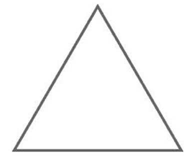

Figure 2: An equilateral triangle has a discrete set of symmetries.

and the Euler-Lagrange gives

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) &= \frac{\partial \mathcal{L}}{\partial x} \\ \frac{d}{dt}(m\dot{x}) &= -kx \\ m\ddot{x} &= -kx \\ \ddot{x} &= \left(\frac{-k}{m}\right)x \end{aligned} \tag{6}$$

This is obviously an equation for simple harmonic motion, so our mass on a spring is a simple harmonic oscillator with the solution $x(t) = A\cos\left(\sqrt{\frac{k}{m}}t - \delta\right)$.

From Example 1.1 we see that the Euler-Lagrange equation results in a differential equation, the solution of which gives our system's equation(s) of motion. With simple systems the differential equation may be easy to solve, but a more complicated system may result in a much more obscure solution. In some situations, Lie symmetries may be applied to solve more difficult equations.

## 2 Lie Symmetries

An academic introduction to group theory will generally include a discussion on the symmetries of geometrical objects. In a simple case, it is easy to see that an equilateral triangle, such as in Figure 2, will have the trivial symmetry, two non-trivial rotations, and three reflections in its symmetric group.

In the case of the equilateral triangle we are examining *discrete symmetries*, because only a certain finite set of discrete rotations and reflections exist within the symmetric group. For example, a rotation of $\frac{2\pi}{3}$ is a symmetry of the equilateral triangle, but rotations of $\pi$ or $\frac{\pi}{6}$ or 2 are not in the symmetric group.

On the other hand, consider a circle. The set of symmetries of a circle is infinite. Any rotation by an angle $\epsilon$ may be represented in Cartesian coordinates as the mapping

$$\Gamma_\epsilon : (x, y) \mapsto (x\cos\epsilon - y\sin\epsilon,\ x\sin\epsilon + y\cos\epsilon) \tag{7}$$

as can be seen in Figure 3. In this case, $\epsilon$ is a continuous parameter. Similarly the set of reflections of a circle can be represented by the mapping

$$\Gamma_\rho : (x, y) \mapsto (-x, y) \tag{8}$$

followed by a rotation $\Gamma_\epsilon$. [2]

Therefore, in the case of a circle we are examining a symmetric group which is not discrete; these kind of symmetries are known as *Lie symmetries*, and they form a *Lie group*, which will be explored in the next section.

Restricting ourselves to ODEs of the form

$$y^{(n)} = \omega(x, y, y', \ldots, y^{(n-1)}), \qquad y^{(k)} \equiv \frac{d^k y}{dx^k}, \tag{9}$$

any symmetry mapping must look like

$$\Gamma\colon (x, y, y', \ldots, y^{(n)}) \mapsto (\hat{x}, \hat{y}, \hat{y}', \ldots, \hat{y}^{(n)}), \qquad \hat{y}^{(k)} = \frac{d^k \hat{y}}{d\hat{x}^k},\ \ k \in \mathbb{N} \tag{10}$$

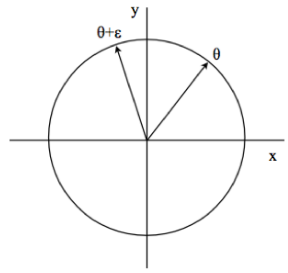


Figure 3: Rotation of a circle by $\epsilon$.

and must fulfill the symmetry condition that

$$\hat{y}^{(n)} = \omega(\hat{x}, \hat{y}, \hat{y}', \dots, \hat{y}^{(n-1)}), \qquad \hat{y}^{(k)} = \frac{d\hat{y}^{(k-1)}}{d\hat{x}} = \frac{D_x \hat{y}^{(k-1)}}{D_x \hat{x}} \tag{11}$$

holds when (10) holds, where $D_x = \partial_x + y'\partial_y + y''\partial_{y'} + \dots$ and $\hat{y}^{(0)} \equiv \hat{y}$. [2]

**Example 2.1** The usefulness of symmetry mappings can be seen visually. In Figure 4, the vector field of some system clearly has no $x$ dependence, so it is invariant in $x$. In another system, whose vector field can be seen in Figure 5, there is clearly symmetry, but it is more difficult to define in terms of $x$ and $y$ than Figure 4 is. However we can apply the symmetry mapping

$$(x, y) \mapsto \left(\sqrt{x^2 + y^2}, \arctan\left(\frac{y}{x}\right)\right) = (r, \theta) \tag{12}$$

and we see that Figure 5 maps to Figure 6. Figure 6 has a much more obvious symmetry – we've "straightened out" the vector field so that it is obviously invariant in $\theta$. We call $r$ and $\theta$ the *canonical coordinates* of the system, and this symmetry is a *canonical transformation*.

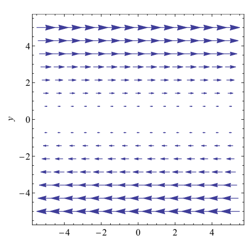

Figure 4: Vector field with $x$ invariance.

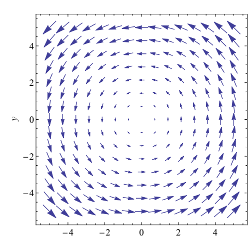

Figure 5: Vector field with rotational symmetry.

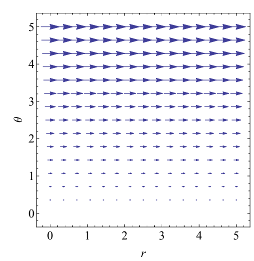

Figure 6: Vector field with $\theta$ invariance.

The ODE that results from the Euler-Lagrange equations in Example 1.1 is easily integrable, so Lie symmetry analysis appears to be superfluous in the solution. The following example from [2] is more complicated.

**Example 2.2** Consider the following ODE:

$$\frac{dy}{dx} = \frac{y^3 + x^2 y - y - x}{xy^2 + x^3 + y - x} \tag{13}$$

This is a nasty looking equation. The solutions are plotted in Figure 7, and there appears to be rotational symmetry, meaning the symmetries

$$(x, y) \mapsto (x\cos\epsilon - y\sin\epsilon, x\sin\epsilon + y\cos\epsilon) \tag{14}$$

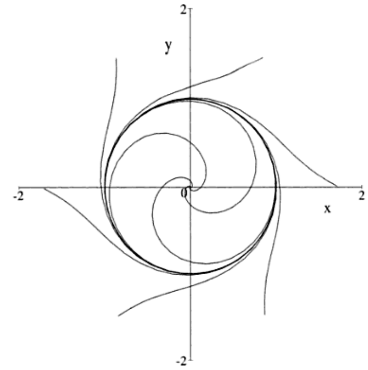


Figure 7: The solutions to (13), from *Symmetry Methods for Differential Equations* by Peter E. Hydon.

form a Lie group for the differential equation. Verifying this would be extremely tedious. However, we can rewrite the differential equation in terms of polar coordinates where $x = r\cos\theta$ and $y = r\sin\theta$. Then (13) becomes

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{\frac{d}{d\theta}[r\sin\theta]}{\frac{d}{d\theta}[r\cos\theta]} \\ &= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}\end{aligned} \tag{15}$$

which we can rearrange to solve for $\frac{dr}{d\theta}$:

$$\begin{aligned}\frac{dr}{d\theta} &= \frac{\frac{-dy}{dx} r\sin\theta - r\cos\theta}{\sin\theta - \frac{dy}{dx}\cos\theta} \\ &= \frac{\frac{-dy}{dx} y - x}{\frac{y}{r} - \frac{dy}{dx}\frac{x}{r}} \\ &= r\left(\frac{\frac{-dy}{dx} y - x}{y - \frac{dy}{dx} x}\right) \\ &= r(1 - r^2)\end{aligned} \tag{16}$$

This ODE is easily integrated, and the symmetries (14) simplify to

$$(r, \theta) \mapsto (r, \theta + \epsilon) \tag{17}$$

which is easily verifiable by applying the symmetry condition (11). This is just a translation in $\theta$, so a vector field for a system with this differential equation may look something like Figure 6; clearly there is no $\theta$ dependence so the system is invariant with respect to $\theta$.

# 3 Lie Groups

**Definition 3.1** A *topological space* is a set $X$ together with a collection of open subsets $T$ that satisfies the following axioms [3]:

i. The empty set is in $T$.

ii. $X$ is in $T$.

iii. The intersection of a finite number of sets in $T$ is also in $T$.

iv. The union of an arbitrary number of sets in $T$ is also in $T$.

**Definition 3.2** A *manifold* is a topological space which is locally Euclidean.

A three-dimensional sphere is an example of a manifold.

**Definition 3.3** Let $U \subset \mathbb{R}^k$, $V \subset \mathbb{R}^n$ be open sets. A map $f\colon U \mapsto V$ is called *smooth* if its every component (and there are $n$) is an infinitely differentiable function.

Our definition of smooth need not always be as strict as to have infinite derivatives; instead, we only need derivatives to exist up to a desired point. For example, if we have a third-order differential equation, we only need derivatives to exist up to fourth order for the function to qualify (for our purposes) as smooth.

**Definition 3.4** A *Lie group* is a group $G$, which is also a manifold, such that the group operations

$$(x, y) \mapsto xy, \qquad x \mapsto x^{-1} \tag{18}$$

are smooth maps from $G \times G$ and $G$, respectively, into $G$ $\forall x, y \in G$.

So, based on our definitions, an element in a Lie group is a continuous differentiable transformation and has an inverse with the same properties. [1]

Because a Lie group is a manifold, it is locally Euclidean. This means that any symmetry parametrized by some sufficiently small $\epsilon$ (in its canonical coordinate system) can be approximated as a translation. This can be done with a Taylor expansion, and leads to the concept of a *Lie algebra*.

# 4 Lie Algebras

**Definition 4.1** A Lie algebra $\mathfrak{g}$ is a vector space over a field $F$ on which a product $[\ ,\ ]$, called the Lie bracket, is defined, with the properties

I. $X, Y \in \mathfrak{g}$ imply $[X, Y] \in \mathfrak{g}$

II. $[X, \alpha Y + \beta Z] = \alpha[X, Y] + \beta[X, Z]$ for $\alpha, \beta \in F$ and $X, Y, Z \in \mathfrak{g}$

III. $[X, Y] = -[Y, X]$

IV. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ [4]

Properties III. and IV are known as skew-symmetry and the Jacobi identity, respectively. A Lie algebra is equivalently defined as the tangent space at the identity of a Lie group, which is much easier to visualize than Definition 4.1.

The Lie bracket may be defined in various ways, as long as it fulfills the conditions of Definition 4.1. A *linear* Lie algebra has matrix elements, where the Lie bracket is defined as the commutator $[X, Y] = XY - YX$. Linear Lie algebras are frequently used in quantum mechanics. Another common example of a Lie algebra is the vector space $\mathbb{R}^3$ with the Lie bracket defined as the cross product, $[X, Y] = X \times Y$ [4].

The basis of a Lie algebra forms the *infinitesimal generators* of its associated Lie group, which are extremely useful in physical applications of finding differential symmetries. Let's go ahead and find the infinitesimal generators of a Lie group; we begin by finding the tangent vectors at any element in the Lie group.

For the Lie group action $(x, y) \mapsto (\hat{x}, \hat{y})$, the tangent vector to $(\hat{x}, \hat{y})$ is $(\xi(x, y), \eta(x, y))$ where

$$\frac{d\hat{x}}{d\epsilon} = \xi(\hat{x}, \hat{y}), \qquad \frac{d\hat{y}}{d\epsilon} = \eta(\hat{x}, \hat{y}).$$

The tangent vector at a point $(x, y)$ is

$$(\xi(x, y), \eta(x, y)) = \left( \left.\frac{d\hat{x}}{d\epsilon}\right|_{\epsilon=0}, \left.\frac{d\hat{y}}{d\epsilon}\right|_{\epsilon=0} \right). \tag{19}$$

So, if we Taylor expand, our Lie group action becomes, to first order in $\epsilon$:

$$\begin{aligned}\hat{x} &= x + \epsilon\xi(x,y) + O(\epsilon^2)\\ \hat{y} &= y + \epsilon\eta(x,y) + O(\epsilon^2)\end{aligned} \tag{20}$$

and our infinitesimal generator $X$ is

$$X = \xi(x,y)\partial_x + \eta(x,y)\partial_y \tag{21}$$

[2].

For this to make more sense, let's do a simple example inspired by [1].

**Example 4.1** Consider a Lie group action $(x,y) \mapsto (\hat{x},\hat{y})$ where

$$\hat{x} = \mathrm{e}^{\epsilon}x^2, \qquad \hat{y} = \mathrm{e}^{-2\epsilon}y. \tag{22}$$

Then we can find our tangent vectors $\xi(\hat{x},\hat{y}), \eta(\hat{x},\hat{y})$:

$$\begin{aligned}\xi(\hat{x},\hat{y}) &= \left.\frac{d\hat{x}}{d\epsilon}\right|_{\epsilon=0} = x^2\\ \eta(\hat{x},\hat{y}) &= \left.\frac{d\hat{y}}{d\epsilon}\right|_{\epsilon=0} = -2y\end{aligned} \tag{23}$$

so our Taylor expansion becomes

$$\hat{x} = x + \epsilon x^2, \qquad \hat{y} = y - 2\epsilon y \tag{24}$$

and our infinitesimal generator is

$$X = x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}. \tag{25}$$

# 5 Noether's Theorem

**Theorem 5.1** *Noether's Theorem* states that any differentiable symmetry of the action of a physical system has a corresponding conservation law.

In other words, "if the Lagrangian function for a physical system is not affected by changes in the coordinate system used to describe them, then there will be a corresponding conservation law." [5]

**Claim 5.1** If the Lagrangian is invariant under a translation of any spatial coordinate, then linear momentum is conserved.

**Claim 5.2** If the Lagrangian is invariant under an angular rotation, then angular momentum is conserved.

**Claim 5.3** If the Lagrangian is invariant under a translation in time, then energy is conserved.

Here I will show a proof, from [5], of Noether's Theorem for conservation of energy.

**Proof.**

Consider a Lagrangian

$$\mathcal{L}[x_i, v_i, t] = T[v_i] - V[x_i] \tag{26}$$

where kinetic energy $T$ is proportional to velocity squared $v_i^2 = \dot{x}_i^2$ and potential energy $V$ is a function only of position $x$.

Consider the first time derivative of (26):

$$\frac{d\mathcal{L}}{dt} = \sum_i \left[ \frac{\partial \mathcal{L}}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial \mathcal{L}}{\partial v_i}\frac{dv_i}{dt} \right] \tag{27}$$

and with the Euler-Lagrange equation (3), this can be written as

$$\begin{aligned} \frac{d\mathcal{L}}{dt} &= \sum_i \frac{d}{dt}\left[ v_i \left( \frac{\partial \mathcal{L}}{\partial v_i} \right) \right] \\ 0 &= \frac{d}{dt}\left[ \sum_i v_i \left( \frac{\partial \mathcal{L}}{\partial v_i} \right) - \mathcal{L} \right] \end{aligned} \tag{28}$$

The time derivative of the expression in square brackets is 0, so that quantity – call it $H$ – must be independent of time; $H$ is conserved in time.

The first term of $H$ is

$$\sum_i v_i \left( \frac{\partial \mathcal{L}}{\partial v_i} \right) \tag{29}$$

but we know that only the kinetic energy term in the Lagrangian is dependent on $v_i$. So (29) is equivalent to

$$\sum_i v_i \left( \frac{\partial T}{\partial v_i} \right). \tag{30}$$

Now we can apply *Euler's Theorem for Homogeneous Functions* [5], which states that for a homogeneous function $f(a_i)$ of degree $k$,

$$kf(a_i) = \sum_i a_i \left( \frac{\partial f(a_i)}{\partial a_i} \right) \tag{31}$$

$T$ is a homogeneous function of order 2, so by (31),

$$2T = \sum_i v_i \left( \frac{\partial T}{\partial v_i} \right) \tag{32}$$

Which means we can write $H$ as

$$\begin{aligned} H &= 2T - \mathcal{L} \\ &= 2T - (T - V) \\ &= T + V. \end{aligned} \tag{33}$$

So $H$ is equal to the kinetic plus potential energies of the system, which is the total energy of the system. Therefore, when the Lagrangian is independent of time, total energy of the system is conserved.

**Example 5.1** Consider Example 2.2. We found that the system described by the ODE

$$\frac{dy}{dx} = \frac{y^3 + x^2 y - y - x}{xy^2 + x^3 + y - x}$$

has rotational symmetry, with the set of symmetries

$$(r, \theta) \mapsto (r, \theta + \epsilon)$$

forming a Lie group. By this symmetry, $\theta$ is invariant, so by Noether's Theorem, angular momentum is conserved in the system.

**Theorem 5.2** Let a Lie group have generators of the form

$$X = \xi_i(x,t)\frac{\partial}{\partial x_i} + \eta_\alpha(x,t)\frac{\partial}{\partial t_\alpha} \tag{34}$$

where the action (2) is stationary. Then the vector

$$T_i = \mathcal{L}\xi_i + (\eta_j - \xi_i t_j)\frac{\partial \mathcal{L}}{\partial t_j} \tag{35}$$

is a conserved vector on the Euler-Lagrange solutions, i.e. $D_i[T_i] = 0$ where $D_i$ is defined as in (11) The proof of this is found in [1].

Now let's tie everything together. In this last example from [1], we will use the generators of a specific Lie algebra called the Lorentz group to analyze the Lagrangian of a free particle in special relativity and conclude the conservation of energy of the system.

**Example 5.2** The Lagrangian of a free particle in special relativity is

$$\mathcal{L} = -mc^2\sqrt{1-\beta^2} \tag{36}$$

where $\beta = \frac{|\dot{\mathbf{x}}|}{c}$ and $|\dot{\mathbf{x}}| = \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2}$, assuming the particle's spatial coordinates are $(x_1, x_2, x_3)$. The generators of the Lorentz group are as follows for $i \in \{1, 2, 3\}$

$$\begin{aligned} X_0 &= \frac{\partial}{\partial t} \\ X_i &= \frac{\partial}{\partial x_i} \\ X_{ij} &= x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j} \\ X_{0i} &= t\frac{\partial}{\partial x_i} + \frac{1}{c^2} x_i \frac{\partial}{\partial t} \end{aligned} \tag{37}$$

Let's focus on $X_0$. By Theorem 5.2,

$$\begin{aligned} T &= \mathcal{L} - \dot{\mathbf{x}}_i \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \\ &= -mc^2\sqrt{1-\beta^2} - \frac{m|\dot{\mathbf{x}}|^2}{\sqrt{1-\beta^2}} \\ &= \frac{-mc^2}{\sqrt{1-\beta^2}} \end{aligned} \tag{38}$$

$T$ is an arbitrarily defined constant, so we can set $T = -E$ and conclude that total energy $E = \frac{mc^2}{\sqrt{1-\beta^2}}$ is conserved.

# 6 Exercises

1. Find the tangent vector at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ for the Lie group action $(x, y) \mapsto (x\cos\phi - y\sin\phi, x\sin\phi + y\cos\phi)$.
2. Consider the following Lagrangian. Which quantities are conserved for this system? Explain.

$$\mathcal{L} = \frac{1}{2}m\dot{y}^2 - mgy$$

# 7 Solutions

1. We have the mapping $(x, y) \mapsto (x\cos\phi - y\sin\phi, x\sin\phi + y\cos\phi) = (\hat{x}, \hat{y})$

$$\begin{aligned} \frac{d\hat{x}}{d\phi} &= \xi(\hat{x}, \hat{y}) = (-x\cos\phi - y\sin\phi) \\ \frac{d\hat{y}}{d\phi} &= \eta(\hat{x}, \hat{y}) = (x\cos\phi - y\sin\phi) \end{aligned}$$

So the tangent vector $(\xi(x, y), \eta(x, y))$ is

$$\left( \frac{d\hat{x}}{d\epsilon}\bigg|_{\epsilon=0}, \frac{d\hat{y}}{d\epsilon}\bigg|_{\epsilon=0} \right) = (-y, x)$$

and at the point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ the tangent vector is evaluated to be $(\frac{-\sqrt{3}}{2}, \frac{1}{2})$.

2. The Euler-Lagrange equation for this Lagrangian is

$$\frac{d}{dt}[m\dot{y}] = -mg$$
$$\cancel{m}\ddot{y} = -\cancel{m}g$$
$$\ddot{y} = -g$$

This ODE doesn't depend on time, position, OR angle. So, energy, linear momentum, and angular momentum are all conserved.

# 8 Acknowledgements

A huge thank you to Professors Greg Elliott and David Latimer for taking the time to discuss Lie groups and conservation laws with me, and for pointing me in the direction of some fantastic sources.

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