Algebraic Graph Theory: Automorphism Groups and Cayley graphs

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1 Introduction

An algebraic approach to graph theory can be useful in numerous ways. There is a relatively natural intersection between the fields of algebra and graph theory, specifically between group theory and graphs. Perhaps the most natural connection between group theory and graph theory lies in finding the automorphism group of a given graph. However, by studying the opposite connection, that is, finding a graph of a given group, we can define an extremely important family of vertex-transitive graphs. This paper explores the structure of these graphs and the ways in which we can use groups to explore their properties.

2 Algebraic Graph Theory: The Basics

First, let us determine some terminology and examine a few basic elements of graphs.

A *graph*, Γ, is simply a nonempty set of vertices, which we will denote $V(\Gamma)$, and a set of edges, $E(\Gamma)$, which consists of two-element subsets of $V(\Gamma)$. If $\{u, v\} \in E(\Gamma)$, then we say that u and v are *adjacent* vertices. It is often helpful to view these graphs pictorially, letting the vertices in $V(\Gamma)$ be nodes and the edges in $E(\Gamma)$ be lines connecting these nodes.

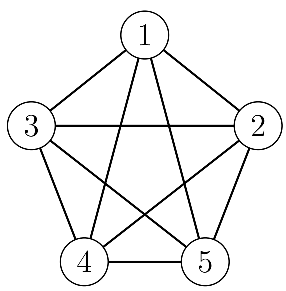
A *digraph*, D is a nonempty set of vertices, $V(D)$ together with a set of ordered pairs, $E(D)$ of distinct elements from $V(D)$. Thus, given two vertices, u, v, in a digraph, u may be adjacent to v, but v is not necessarily adjacent to u. This relation is represented by *arcs* instead of basic edges. The arc set of a digraph, D, is denoted $A(D)$.

It is not uncommon for two graphs to have the same structure. We can examine this situation more carefully by considering *isomorphisms* of graphs. An isomorphism from a graph Γ_1 to Γ_2 is a function ϕ that preserves adjacency; that is, if $\{u, v\} \in E\Gamma_1$, then $\{\phi(u), \phi(v)\} \in E\Gamma_2$. We can of course extend this idea to an isomorphism of a graph Γ to itself, thus leading us to the study of *graph automorphisms*.

Graph automorphisms and groups are closely related structures. A graph automorphism is simply a bijection, $\phi : \Gamma \to \Gamma$, that permutes the the vertices of Γ while preserving adjacency. Let us consider the set of all automorphisms of a graph, Γ. The identity map is of course an element of this set since it preserves adjacency. Given any automorphism, ϕ, ϕ^{-1} is also an automorphisms, and given any two automorphisms ϕ_1, ϕ_2, $\phi_1 \circ \phi_2$ is of course an automorphism. Therefore, the set of all automorphisms of a graph Γ is a group under the operation of function composition. We will denotes this group $\mathrm{Aut}(\Gamma)$ and will commonly refer to it as the group of Γ. Before considering an example, let us first present a definition.

Definition. A graph is *complete* if each vertex is connected to every other vertex. We will denote the complete graph on n vertices by K_n.

Example 1. Below is the connected graph, K_5.



Consider the automorphism group of the above graph. Since all vertices are connected to one another, any permutation of the vertices will preserve adjacency. Therefore, the automorphism group of K_5 is S_5, the set of permutations on 5 elements. It can easily be deduced, then, that the automorphism group of any complete graph, K_n, has automorphism group $\mathrm{Aut}(K_n) = S_n$. Any disconnected graph on n vertices will therefore have an automorphism group that is a subgroup of S_n.

3 Cayley Graphs

As previously explored, given any graph, Γ, we can find an automorphism group, $\mathrm{Aut}(\Gamma)$. This fact, however, raises another question. That is, given any group G, can we find a graphic representation of this group. More specifically, can we find a graph whose group is isomorphic to G. This question leads us to the study of *Cayley Graphs*.

There are three main types of Cayley graphs: Cayley Digraphs, Cayley Color Graphs, and Simple Cayley Graphs. We will define and discuss the first two briefly, while focusing the majority of our attention on the latter of the three.

3.1 Cayley Color Graphs

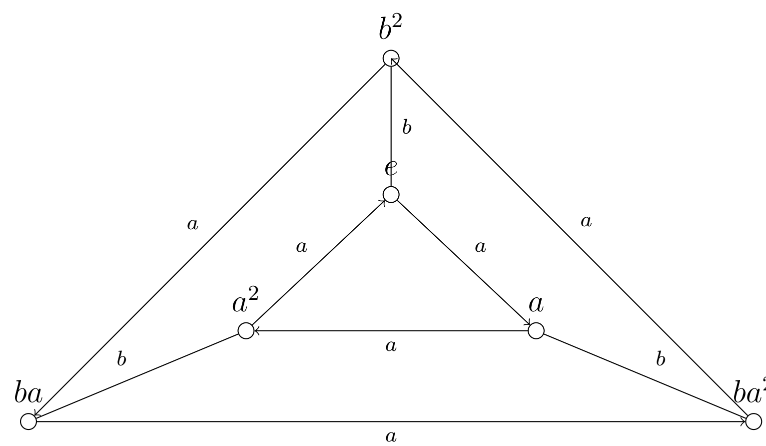
Definition. Let G be a group, and let S be subset of G. Then the Cayley Digraph $D(G,S)$ on G with connection set S is defined as follows:

1. The vertices are the elements of G
2. There is an arc joining g and h if and only if $h = sg$ for some $s \in S$.

We can extend this idea to a Cayley Color graph, where S is a generating set for G, each $s_i \in S$ is assigned a color, and if $g = s_i h$, then the arc connecting them is colored s_i.

Example 2 shows the Cayley Color graph of S_3 with connection set $S = \{a, b\}$ where $a = (123)$ and $b = (12)$. Notice that it is often helpful to represent each group element in terms of group presentations, thus displaying the arcs more clearly.

Example 2. Cayley color graph of S_3 defined on $S = \{(123), (12)\}$.



Now let us examine the automorphism groups of these Cayley color graphs. First, let us present another definition.

Definition. An automorphism $\phi \in \mathrm{Aut}(D(G,S))$ is *color-preserving* if given an arbitrary arc, $\{g,h\}$, $\{g,h\}$ and $\{\phi(g,\phi(h)\}$ have the same color.

Before presenting the main theorem, we must first present an intermediate result.

Proposition 1. *Let G be a group with generating set S and let ϕ be a color-preserving permutation on $V(D(G,S))$. Then ϕ is a color preserving automorphism of $D(G,S)$ if and only if $\phi(gh) = \phi(g)h$.*

Proof. We will prove one side of this theorem, leaving the other as an exercise presented at the end of this paper. (See exercise 2). Suppose that $\phi(gh) = \phi(g)h$. To show that ϕ is color-preserving, we need to show that if $gh^{-1} = s$, then $\phi(gh^{-1}) = s$. Suppose $gh^{-1} = s$. Then

$$\begin{aligned}\phi(gh^{-1}) =& \phi(g)h^{-1}\\ =& \phi(g)g^{-1}s\\ =& \phi(gg^{-1})s\\ =& s.\end{aligned}$$

□

Theorem 2. *Let G be a nontrivial group with generating set S. Then the group of color-preserving automorphisms of $D(G,S)$ is isomorphic to G.*

Proof. Let G be a group of order n and $g_i \in G$ for $1 \leq i \leq n$. Define the map $\phi_i : V(D(G,S)) \to V(D(G,S))$ by $\phi_i(g_) = g_i g$.

This map is surjective, since given any $g \in V(D(G,S))$, $g = \phi_i(g_i^{-1}g)$. This map is also injective since if $\phi_i(g_1) = \phi_i(g_2)$, $g_i g_1 = g_i g_2$ and thus $g_1 = g_2$. Now, let $g_1, g_2 \in G$. By **Proposition 1**, ϕ_i is a color preserving automorphism since

$$\phi_i(g_1 g_2) = g_i(g_1 g_2) = (g_i g_1)g_2 = \phi_i(g_1)g_2$$

Now, let $A = \{\phi_i : 1 \leq i \leq n\}$ and define a map $\alpha : G \to A$ by $\alpha(g_i) = \phi_i$. We must verify that this map is an isomorphism from G to the groups of color preserving automorphisms. This map is injective since $\phi_i \neq \phi_j$ when $i \neq j$. We will show the map is surjective by proving that for any color preserving automorphism ϕ, $\phi = \phi_i$ for some $\phi_i \in A$.

Let e be the identity element of G and let $\phi(e) = g_i$. Given an arbitrary element $g_j \in G$, we can write this element as a product of generators from our generating set S. Let $g_j = s_1^{r_1} s_2^{r_2} ... s_m^{r_m}$. Then by **Proposition 1** we can write,

$$\phi(g) = \phi(eg) = \phi(es_1^{r_1}s_2^{r_2}...s_m^{r_m}) = \phi(e)r_1s_2^{r_2}...s_m^{r_m} = g_ig.$$

Therefore, $\phi = \phi_i$ and α is surjective.

Finally, we must show that α preserves the group operation. Since

$$\phi_{ij}(g) = (g_ig_j)g = g_i(g_jg) = \phi_i(g_jg) = \phi_i(\phi_j(g))$$

we can deduce that,

$$\alpha(g_ig_j) = \phi_{ij} = \phi_i \circ \phi_j = \alpha(g_i) \circ \alpha(g_j)$$

$\square$

Thus, we have shown that given any group G, we can construct a colored digraph representation of this group whose color-preserving automorphism group is isomorphic to G itself. We will now turn our attention to a more generalized version of Cayley Graphs and examine their properties.

3.2 Simple Cayley Graphs

Simple Cayley graphs make up a large amount of graphs in an important family of graphs called *vertex transitive* graphs. They are defined nearly identically as Cayley Digraphs, however they contain an edge set instead of an arc set, and the connection set S must be closed under inverses. More specifically, if $s \in S$, then $s^{-1} \in S$. To further understand these graphs, let us first consider the motivation behind this restriction on S.

Given a Cayley Digraph $D(G, S)$, we know that there exists an arc u, v if and only if $u = sv$. However, if $s^{-1} \in S$, then $v = s^{-1}u$ and therefore v, u is also an arc. Thus, we can replace these two arcs by a simple edge. This restriction on S thus replaces every pair of arcs with an edge, turning our digraph $D(G, S)$ into a simply Cayley graph. We will denote the simple Cayley graph of a group G on the connection set S by $\mathrm{Cay}(G, S)$.

To study these graphs further, let us first review some definitions and theorems about Group Actions.

Recall that given a group G acting on a set V, $x, y \in V$ are *G-equivalent* if $gx = y$ for some $g \in G$. This G-equivalence is an equivalence relation on V. Each partition of V into such an equivalence class is called an *orbit* of V under G. Also recall that the *stabilizer subgroup* of G for an element $x \in V$, denoted G_x, is the set of all group elements in g that fix x. That is, $G_x = \{g \in G : gx = x\}$. Now let us present a new definition.

Definition. A graph Γ is *vertex transitive* if there exists a single orbit of $V(\Gamma)$ under $Aut(\Gamma)$. That is, given any $v, u \in V(\Gamma)$ there exists a $\phi \in \mathrm{Aut}(\Gamma)$ such that $\phi(v) = u$.

Theorem 3. *Every Simple Cayley graph is vertex-transitive*

Proof. Let $\rho_g : v \to vg$ for all $v \in V(\mathrm{Cay}(G, S))$. Clearly, ρ_g permutes the elements of $V(\mathrm{Cay}(G, S))$. To show that $\rho_g \in \mathrm{Aut}(\mathrm{Cay}(G, S))$ we must show that $\{v, u\} \in E(\mathrm{Cay}(G, S))$ if and only if $\{vg, ug\} \in E(\mathrm{Cay}(G, S))$. Suppose $\{v, u\} \in E(\mathrm{Cay}(G, S))$. Then we know that $v = su$ for some $s \in S$ or equivalently, $vu^{-1} = s$. But $vg(ug)^{-1} = vgg^{-1}u^{-1} = vu$. Therefore, $\{v, u\} \in E(\mathrm{Cay}(G, S))$ if and only if $\{vg, ug\} \in E(\mathrm{Cay}(G, S))$ and ρ_g is in the group of $\mathrm{Cay}(G, S)$.

Now, given any vertices, v, u the mapping $\rho_{v^{-1}u}$ maps v to u since $vv^{-1}u = u$. Thus, any Cayley graph is vertex-transitive. □

Just as a subgroup of an automorphism group for a Cayley Color Graph of G contained a subgroup isomorphic to G, simple Cayley graphs possess a similar property. First let us define a bit more terminology.

Definition. A group G acting on a set V is *semiregular* if $G_v = e$ for all $v \in V$. If a group is semiregular and transitive, then we say it is *regular*.

Theorem 4. *Let G be a group and S be an inverse-closed subset of G. Then Aut(Cay(G, S)) contains a regular subgroup isomorphic to G.*

Proof. Let G be group with connection set S and $\mathrm{Cay}(G, S)$ be the Cayley graph for G defined on S. Now, consider the mapping ρ_g as described in Theorem 3. We know that $\rho_g \in \mathrm{Aut}(\mathrm{Cay}(G, S))$, and it can easily be shown that $H = \{\rho_g : g \in G\}$ is a subgroup of G. This group acts regularly on G since it is clearly semiregular, and is also transitive by the proof of **Theorem 3**. The map $\phi : H \to G$ defined by $\phi(\rho_g) = g$ is an isomorphism by Cayley's Theorem (see Judson 9.6). Thus, $H \leq \mathrm{Aut}(\mathrm{Cay}(G, S))$ is isomorphic to G. □

It is natural to ask whether all vertex-transitive graphs are Cayley graphs. This question was negatively answered by the counter-example of the Peterson Graph. The following Theorem gives us criteria for when a graph is indeed a Cayley graph for some group.

Theorem 5. *If a group G acts regularly on the vertices of the graph Γ, then Γ is a Cayley graph for G relative to some inverse-closed connection set S.*

Proof. Let Γ be a graph with degree n and let G be a group that acts normally on $V(\Gamma) = v_1, v_2, ..., v_n$. Since G acts normally on Γ, there exists a unique element $g_i \in G$ such that $g_i v_1 = v_i$. Now, define a set S by

$$S = \{g_i \in G : \{v_i, v_1\} \in E(\Gamma)\}.$$

Now, let x, y be arbitrary elements in $V(\Gamma)$. Then since g_x is an automorphism of Γ, so $\{x, y\} \in E(\Gamma)$ if and only if $\{g_x^{-1}x, g_x^{-1}y\} \in E(\Gamma)$.

Since $g_x u = x$, $g_x^{-1}x = u$. Furthermore, since $g_y u = y$, $g_{x^{-1}} g_y u = g_{x^{-1}} y$. Therefore,

$$g_{x^{-1}} g_y \in S \leftrightarrow \{u, g_{x^{-1}} y\} \in E(\Gamma) \leftrightarrow \{g_{x^{-1}} x, g_{x^{-1}} y\} \in E(\Gamma) \leftrightarrow \{x, y\} \in E(\Gamma).$$

So, if we identify every vertex, x, with the group element g_x, then $\Gamma = Cay(G, S)$. Since Γ is undirected, S is closed under inverses.

□

So, given any graph, Γ and a subgroup G of $\text{Aut}(\Gamma)$, G acts regularly on $V(\Gamma)$ if and only if Γ is a Cayley graph for G for some connection set S.

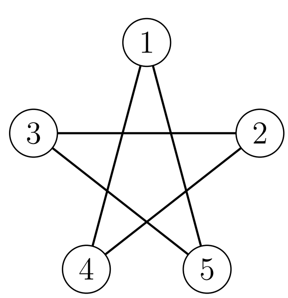
Notice that our definition of a simple Cayley graph does not require our connection set to be a generating set for G. Our next Theorem shows the consequence of S being a generating set for G. First, let us present two definitions.

Definition. A *path* of length r from vertex x to vertex y in a graph is a sequence of $r + 1$ distinct vertices starting with x and ending with y such that consecutive vertices are adjacent.

Definition. A graph, Γ is *connected* if there is a path between any two vertices of Γ.

Theorem 6. *The Cayley graph* $Cay(G, S)$ *is connected if and only if* S *is a generating set for* G.

Example 3. Consider the the group $G = \mathbb{Z}_5$ with connection set $S = \{2, 3\}$. Since $0 + 2^{-1} = 3, 0 + 3^{-1} = 2, 1 + 3^{-1} = 3, 1 + 4^{-1} = 2, 2 + 4^{-1} = 3$, below is the graph, $Cay(G, S)$.



Notice that the graph is connected, since S is a generating set for G.

3.3 Graphical Regular Representation

We know that given a group G and a connection set S, there is a subgroup of Aut(Cay(G, S)) that is isomorphic to G. Often, this subgroup is a proper subgroup of Aut(Cay(G, S)), meaning the entire Aut(Cay(G, S)) is not isomorphic to G. Consider, for example $G = S_3$. When might it be the case that Aut(Cay(G, S)) is itself is isomorphic to G? This question has been successfully answered in the case of finite abelian groups.

Definition. A group G admits *graphical regular representation* if the automorphism group of $Cay(G, S) \cong G$.

Let us again provide an intermediate result before presenting our main theorem for this section.

Proposition 7. *Let $Cay(G, S)$ be a Cayley graph for G defined on the connection set S. Suppose that ϕ is an automorphism of the group G that fixes S set-wise. Then ϕ, regarded as a permutation of the vertices of $Cay(G, S)$ fixes the vertex corresponding to the identity element of G.*

Proof. Let ϕ be a group automorphism. ϕ must fix the the identity element since ϕ is a group automorphism. Let us show that ϕ is a graph automorphism. Suppose that v, w are adjacent vertices. Then vw^{-1} $\in$S.$Therefore, \phi(vw^{-1}) \in S$. But $\phi(vw^{-1}) = \phi(v)\phi(w^{-1})$ so $\phi(v)$ and $\phi(w)$ are adjacent. Therefore, ϕ preserves adjacency and is a graph automorphism.

□

Theorem 8. *Let Γ be a vertex-transitive graph whose automorphism group $G = Aut(\Gamma)$ is abelian. Then G acts regularly on $V(\Gamma)$, and G is an elementary abelian 2-group.*

Proof. Suppose Γ is a vertex-transitive graph with automorphism group $G = \mathrm{Aut}(\Gamma)$. Let $g, h \in G$. Suppose that g fixes an arbitrary vertex $v \in V(\Gamma)$.

$$gh(v) = hg(v) = h(v)$$

Therefore, g fixes $h(v)$ as well. However, since G acts transitively on $V(\Gamma)$ any vertex, u can be written as $h(v) = u$ for some $h \in G$. Thus, g fixes every vertex and the stabilizer is the identity. Therefore, G acts regularly on $V(\Gamma)$ and $\Gamma = Cay(G, S)$ for some connection set S.

Now consider the map $g \to g^{-1}$. This map preserves adjacency since $Aut(\Gamma)$ is abelian and is therefore a graph automorphism. Furthermore, since S is closed under taking inverses, this mapping fixes S set-wise. Thus, by **Proposition 7**, this map must fix vertex 1. But since G acts regularly on $V(\Gamma)$, this map must be the identity map. Therefore, $g = g^{-1}$ and G is an abelian-2 group.

□

Let us think for a moment what this theorem officially states. By **Theorem 5**, we know that if G acts regularly on a graph Γ, then Γ is a Cayley graph for G. Therefore, we know that if a graph has an abelian automorphism group, then this abelian group has a graphical regular representation. Furthermore, we know that this can only happen when G is an abelian 2-group. Thus, the only abelian groups that have a graphical regular representation are abelian-2 groups.

4 Isomorphisms of Cayley Graphs

The final topic we will examine in the study of Cayley Graphs is that of isomorphisms. Determining when Cayley graphs are isomorphic to one another can in fact be a very difficult task. Thus, the bulk of this research has been focused on Cayley graphs of Abelian, or cyclic groups. Due to the complexity of this topic, this paper only briefly examines properties of isomorphic Cayley graphs; however, Beineke, a source listed in the final theorem of this section provide a much more thorough examination of Cayley graph isomorphisms. Let us first present a rather simple proposition regarding isomorphisms of Cayley graph, and then examine more complex criteria for isomorphism.

Proposition 9. *If ϕ is an automorphism of the group G, then $Cay(G, S) \cong Cay(G, \phi(S))$.*

Proof. Given any two vertices, $v, u \in V(\mathrm{Cay}(G, S))$, we know that v and u are connected if and only if $vu^{-1} \in S$. But $\phi(v)\phi(u^{-1}) = \phi(vu^{-1}) \in \phi(S)$ so $\phi(v)$ and $\phi(w)$ are connected if and only if u and v are connected. Therefore, ϕ is an isomorphism from $\mathrm{Cay}(G, S)$ to $\mathrm{Cay}(G, \phi(S))$.

□

Now, let us turn our attention to slightly more complicated situations of isomorphisms between Cayely graphs. First let us present another definition.

Definition. If G is a cyclic group, then the Cayely graph of G defined on some connection set S is called a *circulant.* We will denote these graphs $\mathrm{Circ}(G, S)$.

Now, recall that any cyclic group is isomorphic to $\mathbb{Z}_n$. So, for notational convenience, we will refer to any cyclic group as its isomorphic copy, F_n, and therefore represent any circulant graph in the form, Circ($\mathbb{Z}_n, S$).

Theorem 10. *Let p be a prime. Two circulant graphs Circ($_p, S$) and Circ($_p, S'$) are isomorphic if and only $S' = aS$ for some $a \in \mathbb{Z}_p^*$ where $\mathbb{Z}_p^*$ denotes the multiplicative group of $\mathbb{Z}_p$.*

Proof. We will only prove one side of this theorem, as the other side follows from an application of Burnside's Theorem. (See Theorem 5.1, 5.2 of Beineke). Suppose $S' = aS$ for some $a \in \mathbb{Z}_p^*$. Let ϕ_a be a map from $V(\text{Circ}(\mathbb{Z}, S))$ to $V(\text{Circ}(\mathbb{Z}, S'))$ defined by $\phi_a(v) = av$. Suppose $v, w \in V(\text{Circ}(\mathbb{Z}, S)$. Then v, w are adjacent if and only if $vw^{-1} = s$ for some $s \in S$. But $\phi(vw^{-1}) = \phi(v)\phi(w^{-1}) = avaw^{-1} = avw^{-1} \in aS$. So, v, w are connected if and only if $\phi(v), \phi(w)$ are connected so ϕ preserves adjacency. We also must show that ϕ is both surjective and injective. First let us show surjectivity. If $v \in V(\text{Circ}(\mathbb{Z}_p, S'))$, then since $a \in \mathbb{Z}_p^*$ is a generator for $\mathbb{Z}$ then there exists a $w \in V(\text{Circ}(\mathbb{Z}_p, S))$ such that $v = aw$. Therefore $\phi(w) = v$.

5 Conclusion and Exercises

As we have seen, properties of Cayley graphs have been extensively studied, and this paper only explores some of the many interesting results. Using group structures, we can gain more insight into the structure of these graphs, thus providing classifications of many vertex transitive graphs.

Exercises

1. Provide the Cayley graph for $G = S_3$ defined on the connection set $S = \{(12)\}$.
2. Prove that if $\phi \in H$ where H is the group of color preserving automorphisms of a Cayley color graph, then $\phi(gh) = \phi(g)h$.

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