Mathematics 434 **Spring 2014** **Problem Sheet 7**

Due April 25

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Name

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. **Only write on one side of each page.**

"The road to wisdom? Well it plain and simple to express: Err and err and err again, but less and less and less." -Piet Hein, poet and scientist (1905-1996)

Problems

1. Do both of the following
   (a) Judson Ch 19, #4: Let B be the set of positive integers that are divisors of 36. Define an order on B by $a \preccurlyeq b$ if $a|b$. Prove that B is a Boolean algebra. Find a set X such that B is isomorphic to $\mathcal{P}(X)$.
   (b) Judson Ch 19, #5: Prove or disprove: $\mathbb{Z}$ is a poset under the relation $a \preccurlyeq b$ if $a|b$.
2. Do both of the following
   (a) Judson Ch 19, #15: Let R be a ring and suppose that X is the set of ideals of R. Show that X is a poset ordered by set-theoretic inclusion, $\subseteq$. Define the meet of two ideals I and J in X by $I \cap J$ and the join of I and J by $I + J$. Prove that the set of ideals of R is a lattice under these operations.
   (b) Judson Ch 19, #20: Let X and Y be posets. A map $\phi : X \to Y$ is order-preserving if $a \preccurlyeq b$ implies that $\phi(a) rlt \phi(b)$. Let L and M be lattices. A map $\psi : L \to M$ is a lattice homomorphism if $\psi(a \vee b) = \psi(a) lor \psi(b)$ and $\psi(a \wedge b) = \psi(a) \wedge \psi(b)$. Show that every lattice homomorphism is order-preserving, but that it is not the case that every order-preserving homomorphism is a lattice homomorphism.
3. Do both of the following
   (a) Judson Ch 19, #21: Let B be a Boolean algebra. Prove $a = b$ if and only if $(a \wedge b') \vee (a' \wedge b) = O$ for $a, b \in B$.
   (b) Judson Ch 19, #22: Let B be a Boolean algebra. Prove $a = 0$ if and only if $(a \wedge b') \vee (a' \wedge b) = b$ for all $b \in B$.
4. Judson Ch 19, #16: Let B be a Boolean algebra. Prove each of the following identities.
   (a) $a \vee I = I$ and $a \wedge O = O$ for all $a \in B$.
   (b) If $a \vee b = I$ and $a \wedge b = O$, then $b = a'$.
   (c) $(a')' = a$ for all $a \in B$.
   (d) $I' = O$ and $O' = I$.
   (e) $(a \vee b)' = a' \wedge b'$ and $(a \wedge b)' = a' lor b'$ (De Morgans laws).
5. Judson Ch 19, #10: (See page 321 of the textbook for the pictures of the three circuits.)

   For each of the following circuits, write a Boolean expression. If the circuit can be replaced by one with fewer switches, give the Boolean expression and draw a diagram for the new circuit.

6. Do both of the following:

   (a) Let F be a field. Find all elements a such that $a = a^{-1}$

   (b) Let R be an integral domain containing a field F as subring and which is finite-dimensional when viewed as a vector space over F. Prove that R is a field.

7. Let F be a field containing exactly 8 elements. Prove or disprove that the characteristic of F is 2.

8. Let α be the real cube root of 2. What is the irreducible polynomial for $1 + \alpha^2$ over $\mathbf{Q}$?

9. Determine the irreducible polynomial for $\alpha = \sqrt{3} + \sqrt{5}$ over each of the following fields:

   (a) $\mathbf{Q}$

   (b) $\mathbf{Q}\left(\sqrt{5}\right)$

   (c) $\mathbf{Q}\left(\sqrt{10}\right)$

   (d) $\mathbf{Q}\left(\sqrt{15}\right)$

10. Let α be a complex root of the irreducible (in $\mathbf{Q}$) polynomial $x^3 - 3x + 4$. Find the inverse of $\alpha^2 + \alpha + 1$ in $F(\alpha)$. Write your answer in the form $a + b\alpha + c\alpha^2$ where $a, b, c \in \mathbf{Q}$.