Mathematics 434 Spring 2014 Problem Sheet 3

Due March 7: Problem Sheets 1, 2, 3

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Name

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. **Only write on one side of each page.**

"*all ignorance toboggans into know and trudges up to ignorance again.*" – e.e.cummings, 1959

Problems

1. **You Must do this problem.**

   Let I, J be ideals of a ring R.

   (a) Show by example that $I \cup J$ need not be an ideal but show the set $I+J = \{r \in R : r = x + y,\ x \in I,\ y \in J\}$ is an ideal. This ideal is called the **sum** of I and J.

   (b) Prove that $I \cap J$ is an ideal.

   (c) Show by example that the set of products $\{xy : x \in I,\ y \in J\}$ need not be an ideal but that the set of finite sums $\sum_{i,j} x_i y_j$ of products of elements of I and J is an ideal. This ideal is called the **product** ideal and is denoted IJ.

   (d) Prove $IJ \subset I \cap J$.

   (e) Show by example that IJ and $I \cap J$ need not be equal.

2. Do both of the following

   (a) For which integers n does $x^2 + x + 1$ divide $x^4 + 3x^3 + x^2 + 6x + 10$ in $(\mathbb{Z}/n\mathbb{Z})[x]$?

   (b) Describe the kernel of the map defined by $\phi : \mathbb{Z}[x] \to \mathbb{R}$ given by $\phi(f(x)) = f\left(1 + \sqrt{2}\right)$.

3. Do both of the following

   (a) Prove the kernel of the homomorphism $\phi : \mathbb{C}[x, y] \to \mathbb{C}[t]$ given by $\phi(f(x, y)) = f\left(t^2, t^3\right)$ is the principal ideal generated by the polynomial $y^2 - x^3$.

   (b) Describe the image of ϕ explicitly.

4. Prove that every nonzero ideal in the ring of Gauss integers contains a nonzero integer.

5. Let R be a commutative ring with identity of characteristic p where p is a prime. Prove that if a is nilpotent in R then $1 + a$ is unipotent, that is, some power of $1 + a$ is equal to 1. [An element in a ring is **nilpotent** if some positive integer power of the element equals 0.]

6. Do **one** of the following.

   (a) Determine the structure of the ring $\mathbb{Z}[x] / \left(x^2 + 3, p\right)$ where

      i. $p = 3$

      ii. $p = 5$

   (b) Describe the ring $\mathbb{Z}[i] / (2 + i)$ in terms of "more familiar" rings.

7. Describe the ring obtained from $\mathbb{Z}$ by adjoining an element α satisfying the two relations $2\alpha - 6 = 0$ and $\alpha - 10 = 0$.

8. Suppose we adjoin an element α to $\mathbb{R}$ satisfying the relation $\alpha^2 = 1$. Prove the resulting ring is isomorphic to the product ring $\mathbb{R} \times \mathbb{R}$,and find the element of $\mathbb{R} \times \mathbb{R}$ which corresponds to α.

9. Let α denote the residue of x in the ring $R' = \mathbb{Z}[x] / \left(x^4 + x^3 + x^2 + x + 1\right)$. Compute the expressions for $\left(\alpha^3 + \alpha^2 + \alpha\right)(\alpha + 1)$ and α^5 in terms of the basis $\left(1, \alpha, \alpha^2, \alpha^3, \alpha^4\right)$.

10. Do **one** of the following.

    (a) In each case describe the ring obtained from $\mathbb{Z}$ by adjoining an element α satisfying the given relation.

      i. $\alpha^2 + \alpha + 1 = 0$

      ii. $\alpha^2 + 1 = 0$

    (b) Let $R = \mathbb{Z}/(10)$. Determine the structure of the ring R' obtained from R by adjoining element α satisfying each relation.

      i. $2\alpha - 6 = 0$

      ii. $2\alpha - 5 = 0$.

11. Describe the ring obtained from $\mathbb{Z}/12\mathbb{Z}$ by adjoining an inverse of 2.In particular, what 'standard' ring is isomorphic to this adjunction ring?

12. Let a be an element of a ring R, and let $R' = R[x]/(ax - 1)$ be the ring obtained by adjoining an inverse of a to R. Prove that the kernel of the canonical map from R to R' is the set of elements $b \in R$ such that $a^n b = 0$ for some $n > 0$.

13. Let F be a field, t a free symbol and $R = F[t]$ the ring of polynomials on the variable t with coefficients in F. Adjoin an inverse to the variable t by forming the quotient $F[t,x]/(xt - 1)$. Show that this ring is isomorphic to the ring $F\left[t, t^{-1}\right]$ of Laurent polynomials. A Laurent polynomial is a polynomial in t and t^{-1} of the form

$$f(t) = \sum_{i=-n}^{n} a_i t^i = a_{-n}t^{-n} + \cdots a_{-1}t^{-1} + a_0 + a_1 t + \cdots + a_n t^n.$$