Mathematics 434 Spring 2014 Problem Sheet 5

Due May 4

Name

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. **Only write on one side of each page.**

"Drawing on my fine command of the English language, I said nothing." — Robert Benchley

Problems

Some of these problems might be duplicates of previous sheets. You may not have credit for the same problem twice.

1. Let a, b be elements of a field F with $a \neq 0$. Prove a polynomial $f(x) \in F[x]$ is irreducible if and only if $f(ax+b)$ is irreducible.

2. Prove the kernel of the evaluation homomorphism $\phi : \mathbf{Z}[x] \to \mathbf{R}$ with $\phi(f(x)) = f\left(1+\sqrt{2}\right)$ is a principal ideal and find a generator for this ideal.

3. Prove the following polynomials are irreducible in $\mathbf{Q}[x]$

   - $x^2 + 27x + 213$, $8x^3 - 6x + 1$, $x^5 - 3x^4 + 3$

4. Factor $x^5 + 5x + 5$ into irreducible factors in $\mathbf{Q}[x]$ and in $(\mathbf{Z}/2\mathbf{Z})[x]$.

5. Using reduction modulo 2 as an aid, factor the following polynomials in $\mathbf{Q}[x]$

   (a) $x^2 + 2345x + 125$
   (b) $x^4 + 2x^3 + 2x^2 + 2x + 2$
   (c) $x^4 + 2x^3 + 3x^2 + 2x + 2$
   (d) $x^5 + x^4 - 4x^3 + 2x^2 + 4x + 1$

6. Factor the following into primes in $\mathbf{Z}[i]$.

   - 30, $1-3i$, 10, $6+9i$

7. Prove that a polynomial with integer coefficients is primitive if and only if it is not contained in any of the kernels of the substitution homormorphisms $h_p : \mathbf{Z}[x] \longrightarrow \mathbf{Z}/p\mathbf{Z}[x]$ that takes $f(x) = \sum a_n x^n$ to the polynomial $\sum \bar{a}_n x^n$ where p is a prime integer and $\bar{a}$ denotes the equivalence class of a in $\mathbf{Z}/p\mathbf{Z}$.

8. Prove that two integer polynomials are relatively prime in $\mathbf{Q}[x]$ if and only if the ideal they generate in $\mathbf{Z}[x]$ contains an integer.

9. Prove that $\det \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is irreducible in the polynomial ring $\mathbf{C}[x, y, z, w]$.