A Basic Introduction to Hopf Algebras

Becca Ebert

April 18, 2014

Abstract

This paper offers an introduction to Hopf algebras. It is not a paper describing basic properties and applications of Hopf algebras. Rather, it is a paper providing the necessary structures in order to construct a Hopf algebra. It examines tensor products, algebras, coalgebras, bialgebras, and homology groups. After reading this paper, the reader will not be sufficiently experienced in Hopf algebras to pursue complex applications. However, the reader will be well-versed in the underlying structure of Hopf algebras, which will help any future applications.

1 Introduction

Hopf algebras are an important topic of study in mathematics because of their wide range of applications. Hopf algebras were originally used in topology in the 1940s, but since then their applications and popularity as a topic of study have grown tremendously. Hopf algebras are used in combinatorics, category theory, homological algebra, Lie groups, topology, functional analysis, quantum theory, and Hopf-Galois theory. While exploring these applications is beyond the scope of this paper, this paper attempts to prepare the reader for further study using Hopf algebras.

This paper draws extensively from definitions to construct the underlying structures and properties of Hopf algebras and culminates in the definition and an example of a Hopf algebra. The purpose of this is to create a very solid foundation on which to build further studies in Hopf algebras. This paper also repeatedly uses tensor products and commutative diagrams, which are important tools for continuing studies in mathematics. The goal is for the reader to gain a deep understanding of the underlying structure of Hopf algebras as well as recognition and curiosity of the more complex and convoluted aspects of Hopf algebras.

2 Preliminaries

The study of Hopf algebras draws heavily on concepts including groups, fields, vector spaces, morphisms, and commutative diagrams. This section provides definitions of these concepts to use as a reference for later material.

Definition 2.1 (**Group**)**.** A group $(G, \circ)$ is a set G with a defined binary operation $\circ$ in which

1. $\circ$ is associative.
2. G has an identity element, e.
3. Each element a has an inverse a^{-1} such that $a \circ a^{-1} = a^{-1} \circ a = e$.

Additionally, an abelian group is one in which $a \circ b = b \circ a$ for every $a, b \in G$.

Definition 2.2 (**Field**)**.** A field $\mathbb{K}$ is a commutative ring with identity where every element $\alpha \in \mathbb{K}$ has an inverse.

Definition 2.3 (**Vector Space**)**.** A Vector space V over a field $\mathbb{K}$ is an abelian group with a scalar product defined for all $f \in K$ and all $v \in V$ satisfying the following axioms where $\alpha, \beta \in \mathbb{K}$ and $u, v \in V$:

1. $\alpha(\beta v) = (\alpha\beta)v$
2. $(\alpha + \beta)v = \alpha v + \beta v$
3. $\alpha(u + v) = \alpha u + \alpha v$
4. $1v = v$

The $\alpha, \beta \in \mathbb{K}$ are called scalars while the $u, v \in V$ are the vectors. A vector space with scalars coming from a field $\mathbb{K}$ is called a $\mathbb{K}$-vector space.

Definition 2.4 (**Homomorphism**)**.** A homomorphism is a map between two objects each defined with a binary operation where the map preserves these operations. That is, for two objects with associated operations, $(X, \bullet)$ and $(Y, \circ)$, a homomorphism is a map $\phi : X \to Y$ where for $x_1, x_2 \in X$,

$$\phi(x_1 \bullet x_2) = \phi(x_1) \circ \phi(x_2).$$

This paper will use the term morphism to avoid the need for further clarification unless it is imporant to indicate whether a mapping is a homomorphism, automorphism, or isomorphism.

Definition 2.5 (**Commutative Diagram**)**.** A commutative diagram is a diagram showing the composition of maps represented by arrows. The commutativity of the diagram requires that no matter which direction through the diagram one takes, the result of the series of map compositions will be the same.

The use of commutative diagrams in widespread in the study of Hopf algebras. However, it is assumed the reader has already had experience with a commutative diagram since the diagram often used to represent the First Isomorphism Theorem is a commutative diagram. This was a brief definition of a commutative diagram, but this paper makes extensive use of these diagrams in its definitions. Understanding their properties comes with increased familiarity, which is one of the goals of this paper.

These definitions are important to keep in mind as the background for the structure of Hopf Algebras. We will now move on to a fundamental operation of Hopf algebras, the tensor product.

3 Tensor Products

This section offers an introduction to tensor products as necessary for its use in Hopf algebras. Therefore the theorems and properties in this section were specifically chosen as those pertinent to understanding the definition of a Hopf algebra. Consequently, many important properties are not included. However, the information included is important in understanding the structures and operations within Hopf algebras. In its most basic sense, a tensor product is a multiplication of vector spaces V and W resulting in a single vector space, $V \bigotimes W$ (read V tensor W). We begin with a basic definition of this multiplication.

Definition 3.1. Let V and W be $\mathbb{K}$-vector spaces with bases $\{e_i\}$ and $\{f_j\}$ respectively. The tensor product of V and W is a new $\mathbb{K}$-vector space, $V \bigotimes W$ with a basis $\{e_i \otimes f_j\}$. Notationally,

$$V \bigotimes W \text{ is the set of all elements } v \otimes w = \sum_{i,j} c_{i,j} e_i \otimes f_j$$

Where $c_{i,j} \in \mathbb{K}$ are scalars.

The tensor product does not depend on the choice of basis. In fact, this definition can be generalized to avoid bases and instead require only a spanning set. In this case, the elements, $v \otimes w \in V \bigotimes W$ span $V \bigotimes W$ and are subject to the following properties.

Proposition 3.2. Let the set of $v \otimes w \in V \bigotimes W$ be a spanning set of $V \bigotimes W$. Each $v \otimes w$ must satisfy the following laws:

1. $(v + v') \otimes w = v \otimes w + v' \otimes w$
2. $v \otimes (w + w') = v \otimes w + v \otimes w'$
3. For $\alpha \in \mathbb{K}$, $\alpha(v \otimes w) = \alpha v \otimes w = v \otimes \alpha w$

Therefore, the tensor product upholds distributive laws and laws of scalar multiplication. These properties are central to what it means to be bilinear, which will be defined later.

Note that elements in $V \bigotimes W$ are rarely what are called pure tensors, which have the form $v \otimes w$. Rather, they are linear combinations of pure tensors.

Example 3.3. Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$. Choose the standard basis vectors as the basis for V and for W i.e. the basis of V=$\{e_1,\ e_2\}$ and the basis of W=$\{f_1,\ f_2,\ f_3\}$. The basis of $\mathbb{R}^2 \bigotimes \mathbb{R}^3$ is $\{e_1 \otimes f_1,\ e_1 \otimes f_2,\ e_1 \otimes f_3,\ e_2 \otimes f_1,\ e_2 \otimes f_2,\ e_2 \otimes f_3\}$.

Notice that the dimension of $\mathbb{R}^2 \bigotimes \mathbb{R}^3$ is six, which is the product of the dimension of V and of W. This brings us to our second proposition.

Proposition 3.4. The dimension of the tensor product of two vector spaces is equal to the product of the dimensions of the two vector spaces. That is,

$$dim(V \bigotimes W) = dim(V)dim(W)$$

Recall that the Cartesian Product of two vector spaces, V and W, is the set of all ordered pairs with elements in each of the vector spaces. $V \times W = \{(v, w) : v \in V, w \in W\}$. The dimension of $V \times W$ is equal to the sum of the dimensions of V and W. Comparing this to the tensor product, the resulting vector space is much larger than that of the Cartesian Product. While the Cartesian Product and the tensor product are vastly different, there is an important relationship between them, which is defined in our next theorem.

Theorem 3.5 (**Universal Property of Tensor Products**). *Let V, W and Y be vector spaces. And the map $f : V \times W \to Y$ is defined as $f : (v, w) \mapsto vw$.*

1. *There exists a bilinear mapping α defined as $\alpha : V \times W \to V \bigotimes W$ where $(v, w) \mapsto v \otimes w$.*
2. *If $f : V \times W \to Y$ is bilinear, then there is a unique linear function, $\hat{f} : V \otimes W \to Y$ with $f = \hat{f} \circ \alpha$*

That is, the following diagram commutes:

$$\begin{tikzcd}
V \times W \arrow[r, "\alpha"] \arrow[dr, "f"] & V \otimes W \arrow[d, "\hat{f}"] \\
& \mathrm{Y}
\end{tikzcd}$$

Figure 1: Universal Property of Tensor Products

Note that a bilinear function is one where the mapping is linear in each variable when the other is fixed, which is what the properties of the tensor product in Proposition 3.2 indicate.

The commutativity of this diagram means that the result of $(\hat{f} \circ \alpha)(v, w)$ equals $f(v, w)$. The commutative diagrams will get more complicated, so it is important to understand what this diagram represents before more mappings are included.

It is important to note that this $\hat{f}$ is precisely the multiplication in algebras that will be defined in Section 4.1.

Proposition 3.6. In $V \bigotimes W$, for $v \in V$ and $w \in W$, the following are true:

1. $0 = 0 \otimes 0$
2. $0 = 0 \otimes w$
3. $0 = v \otimes 0$

These are useful facts when computing tensor products and proving theorems such as the next theorem. The general outline of this proof was taken from [1].

Theorem 3.7. *The tensor product of the complex numbers and the ring of polynomials with real coefficients is isomorphic to the ring of polynomials with complex coefficients. That is,*

$$\mathbb{C} \bigotimes \mathbb{R}[x] \cong \mathbb{C}[x].$$

where $\mathbb{C}$ and $\mathbb{R}[x]$ are $\mathbb{R}$-vector spaces.

Proof. Define a map $\alpha : \mathbb{C} \times \mathbb{R}[x] \to \mathbb{C}[x]$ by $\alpha(z, p(x)) \mapsto zp(x)$ where $z \in \mathbb{C}$ and $p(x) \in \mathbb{R}[x]$.
Let $z, z' \in \mathbb{C}$ and $p(x), q(x) \in \mathbb{R}[x]$ and $\gamma \in \mathbb{R}$ be a scalar.

Consider,

$$\alpha(z, p(x) + q(x)) = z(p(x) + q(x)) = zp(x) + zq(x)$$
$$\alpha(z + z', p(x)) = (z + z')p(x) = zp(x) + z'p(x)$$
$$\alpha(\gamma z, p(x)) = \gamma zp(x) = z\gamma p(x) = \alpha(z, \gamma p(x))$$

Since α respects distributive and scalar multiplication laws when one element is fixed, α is a bilinear morphism.

Consider the morphism $\hat{f} : \mathbb{C} \bigotimes \mathbb{R}[x] \to \mathbb{C}[x]$ where $\hat{f}(z \otimes p(x)) \mapsto zp(x)$.

To check if $\hat{f}$ is surjective, check if an element maps to each element in the basis of $\mathbb{C}[x]$. Use $\{ix^n, x^n\}$ as a basis.

$$\hat{f}(i \otimes x^n) \mapsto ix^n$$
$$\hat{f}(1 \otimes x^n) \mapsto x^n$$

Since there exists an element in the domain that maps to each element in the basis, $\hat{f}$ is surjective.

Now suppose

$$\hat{f}\left(\sum_k z_k \otimes p_k(x)\right) = 0$$
$$\hat{f}\left(\sum_k (a_k + b_k i) \otimes p_k(x)\right) = 0$$
$$\sum_k (a_k + b_k i)p_k(x) = 0$$
$$\sum_k a_k p_k(x) + \sum_k b_k i p_k(x) = 0$$

So, $\sum_k a_k p_k(x) = 0$ and $\sum_k b_k i p_k(x) = 0$.

Now consider an arbitary real element in $\mathbb{C} \bigotimes \mathbb{R}[x]$. Call it $\sum_k a_k \otimes p_k(x)$. Using the distributive and scalar laws of tensor products and the conclusions above, you get

$$\begin{aligned} \sum_k a_k \otimes p_k(x) &= \sum_k 1 \otimes a_k p(x) \\ &= 1 \otimes \sum_k a_k p_k(x) \\ &= 1 \otimes 0 \\ &= 0 \end{aligned}$$

Similarly, consider an arbitrary imaginary element in $\mathbb{C} \bigotimes \mathbb{R}[x]$ call it $\sum_k b_k i \otimes p_k(x)$.

$$\begin{aligned} \sum_k b_k i \otimes p_k(x) &= \sum_k i \otimes b_k p_k(x) \\ &= i \otimes \sum_k b_k p_k(x) \\ &= i \otimes 0 \\ &= 0 \end{aligned}$$

By combining the real and imaginary components, any element in $\mathbb{C} \bigotimes \mathbb{R}[x]$ that maps to 0 is the element $\sum_k (a_k + b_k i) \otimes p_k(x) = 0$.
Therefore, the $ker(\hat{f}) = 0$. Thus, $\hat{f}$ is injective.
Since $\hat{f} : \mathbb{C} \bigotimes \mathbb{R}[x] \to \mathbb{C}[x]$ is both injective and surjective, it is an isomorphism.
Therefore,

$$\mathbb{C} \bigotimes \mathbb{R}[x] \cong \mathbb{C}[x].$$

□

The Universal Property of Tensor Products is an important tool for determining isomorphisms involving tensor products.

Proposition 3.8 (**Extension of Tensor Products**)**.** The definition of a tensor product can naturally be extended to more than two vector spaces. Let V_i be a vector space with basis $\{v_i\}$. Then,

$$V_1 \bigotimes V_2 \bigotimes \cdots \bigotimes V_n = \sum_i c_i v_1 \otimes v_2 \otimes \cdots \otimes v_n$$

It is not expected that the reader now has an extensive understanding of tensor products. In fact, at this point it is not even expected that the reader knows exactly what a tensor is. However, it is expected that the reader, when confronted with a tensor product, has a basis from which to begin to discover how it is functioning on a very rudimentary level and to have the tools to do basic manipulations, which will be increasingly important as we go on. This is important since tensor products are central in defining algebras, coalgebras, and bialgebras, which we do next.

4 Bialgebras

Any study of Hopf algebras must be rooted in the study of algebras in general. A Hopf algebra is a bialgebra, so it has both an algebraic and coalgebraic structure. The basic properties of algebras and coalgebras are explored in this section.

4.1 Algebras

An algebra is simultaneously a vector space and a ring. That is, it has all the properties of a vector space while also having two closed binary operations necessary for a ring. In the algebras studied with Hopf algebra, the operations are an associative multiplication, m, and a unit, u.

Definition 4.1. Given a field $\mathbb{K}$ and a vector space A over the field $\mathbb{K}$, with $m : A \otimes A \to A$ and $u : \mathbb{K} \to A$, (A, m, u) is a $\mathbb{K}$-algebra if the following diagrams commute:

$$\begin{tikzcd}
 & A \otimes A \otimes A \arrow[dl, "m\otimes 1"'] \arrow[dr, "1\otimes m"] & \\
A \otimes A \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\
 & A &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes \mathbf{k} \arrow[d, "1\otimes u"'] & A \arrow[l] \arrow[r] \arrow[d, "1"] & \mathbf{k} \otimes A \arrow[d, "u\otimes 1"] \\
A \otimes A \arrow[r, "m"] & A & A \otimes A \arrow[l, "m"']
\end{tikzcd}$$

Figure 2: Association and unit of algebras

Note that u is the unit mapping that sends elements of $\mathbb{K}$ to A where A is both a vector space and a ring. The map u sends the multiplicative identity in $\mathbb{K}$ to a multiplicative identity in the ring, 1_A, making the ring unitary. Additionally, $u(k) \mapsto k(1_A)$, so u is a mapping that transfers elements in $\mathbb{K}$ to A by using the multiplicative identity in the ring, A.

Additionally, $m \otimes 1 : A \bigotimes A \bigotimes A \to A \bigotimes A$ is a type of function composition. To generalize, let f be a generic mapping, 1 be an identity mapping, and let a, b, c be elements of the vector spaces A, B, and C respectively. Then the function composition is,

$$(f \otimes 1)(a \otimes b \otimes c) = f(a \otimes b) \otimes 1(c).$$
$$(1 \otimes f)(a \otimes b \otimes c) = 1(a) \otimes f(b \otimes c).$$

Proposition 4.2. The commutativity of the first diagram demonstrates the associativity of the algebra's multiplication.

Proof. Think of the diagram elementwise.
Start with $a \otimes b \otimes c \in A \bigotimes A \bigotimes A$.
$(m \otimes 1)(a \otimes b \otimes c) \mapsto m(a \otimes b) \otimes c$
Following the diagram, $m(m(a \otimes b) \otimes c) = m(ab \otimes c) = abc \in A$

Similarly, start with $a \otimes b \otimes c \in A \bigotimes A \bigotimes A$ but follow the diagram through the other direction.
$(1 \otimes m)(a \otimes b \otimes c) \mapsto a \otimes m(b \otimes c)$
Continuing through the diagram, $m(a \otimes m(b \otimes c)) = m(a \otimes bc) = abc \in A$

Since $abc = abc$, this diagram commutes and $m(m(a \otimes b) \otimes c) = m(a \otimes m(b \otimes c))$
which shows m is associative.

□

Proposition 4.3. The commutativity of the second diagram requires the existence of a multiplicative identity.

Proof. By an argument similar to the one above, think of the diagram elementwise.
Start with $k \otimes a \in \mathbb{K} \bigotimes A$.
$k \otimes a = 1k \otimes a = 1 \otimes ka = 1 \otimes a'$
$(u \otimes 1)(1 \otimes a') = u(1) \otimes a'$.
Following the diagram, $m(u(1) \otimes a') = u(1)a'$.

Similarly, start with $a \otimes k \in A \bigotimes \mathbb{K}$ but follow the diagram through the other direction.
$a \otimes k = a \otimes 1k = ak \otimes 1 = a' \otimes 1$.
$(1 \otimes u)(a' \otimes 1) = a' \otimes u(1)$.
Following the diagram, $m(a' \otimes u(1)) = u(1)a'$.

For this diagram to commute, $u(1)a' = a'u(1) = a'$
Thus $u(1)$ is the identity element. □

With this definition and proposition, let us now turn to an example of an algebra.

Theorem 4.4. *If $\mathbb{K}$ is a field and G is an abelian group, let*

$$\mathbb{K}G = \{\sum_{i=1}^{n} \mathbb{K}, g \in G\}$$

with multiplication defined as $(\alpha g)(\beta h) = (\alpha\beta)(gh)\ \forall \alpha, \beta \in \mathbb{K}$ and $g, h \in G$ and where

$$m : \mathbb{K}G \bigotimes KG \to \mathbb{K}G \ \textit{by}\ m(g \otimes h) \mapsto gh$$
$$\textit{and}\ u : \mathbb{K} \to \mathbb{K}G \quad \textit{by} \quad u(k) \mapsto 1_{\mathbb{K}G}.$$

Then $\mathbb{K}G$ is an algebra over the field $\mathbb{K}$.

Proof. To prove $\mathbb{K}G$ is an algebra, we must show m is associative and KG has a multiplicative identity.

To begin, consider $m(\alpha g \otimes 1_{\mathbb{K}} 1_G) = \alpha 1_{\mathbb{K}} g 1_G = \alpha g$.

Similarly, $m(1_{\mathbb{K}} 1_G \otimes \alpha g) = 1_{\mathbb{K}} \alpha 1_G g = \alpha g$.

So, $1_{\mathbb{K}}1_G$ is the multiplicative identity in $\mathbb{K}G$.

To show m is associative, let $g, h, j \in \mathbb{K}G$.
Consider,

$$\begin{aligned} m(g \otimes m(h \otimes j)) &= m(g \otimes hj) \\ &= ghj \\ &= m(gh \otimes j) \\ &= m(m(g \otimes h) \otimes j) \end{aligned}$$

So, m is associative.
Since m is associative, and $\mathbb{K}G$ has a multiplicative identity, $1_{\mathbb{K}}1_G$, the diagrams commute, and $\mathbb{K}G$ is an algebra over the field $\mathbb{K}$. $\square$

The algebraic structure has a natural complement, the coalgebra. A coalgebra has its own mappings, but it has a similar commutative diagram that is the algebraic diagram with all the arrows reversed. Coalgebras are explored more explicitly in the next section.

4.2 Coalgebras

A coalgebra is imbued with a comultiplication, Δ, and a counit, ϵ, that have natural relationships to the multiplication and unit of algebras. These specific operations will be explained following the definition of a coalgebra.

Definition 4.5. Given a field $\mathbb{K}$ and a vector space C over the field $\mathbb{K}$, with $\Delta : A \to A \otimes A$ and $\epsilon : A \to \mathbb{K}$, (C, Δ, ϵ) is a $\mathbb{K}$-coalgebra if the following diagrams commute:

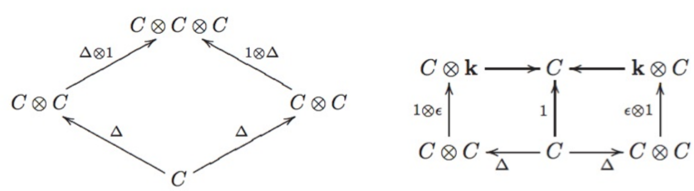

Figure 3: Coassociation and counit of coalgebras

Notice that these diagrams are the same as those for an algebra but with the arrows reversed. This demonstrates the dual nature of algebras and coalgebras. And similar to algebras, the commutativity of the first diagram demonstrates the coassociativity of the comultiplication and the commutativity of the second demonstrates the multiplicative identity, the counit. The proof of these mirror the proofs for associativity and units of algebras and is left to the reader as an exercise.

Comultiplication can be challenging to conceptualize. If multiplication is thought of as taking two elements and merging them into one, comultiplication has the opposite effect by taking one element dividing it, and in the case of the tensor product, obtaining a finite series of pairs of elements. This can often be messy notationally, so the Sweedler Notation is used as a shorthand way of writing tensor products.

Definition 4.6 (Sweedler Notation)**.** For an element c in a coalgebra C,

$$\Delta(c) = \sum_{(c)} c_1 \otimes c_2$$

This notation can be extended for greater than two vector spaces. In this case we write,

$$\Delta(c) = \sum_{(c)} c_1 \otimes \cdots \otimes c_{n+1}$$

For any single element, c in a coalgebra, C, we can write it as the tensor of at least two elements. For computations, it is not necessary to know the exact elements in the tensor product. Rather, it is often necessary to only denote that there are multiple elements. For example, an element $c_{ij} \in C$ can be written as,

$$\Delta(c_{ij}) = \sum_{(c_{ij})} c_i \otimes c_j.$$

This allows for tracking the specific elements and is particularly useful when proving theorems and properties of tensor products. However, it is not necessary for the following theorem.

Theorem 4.7. *If $\mathbb{K}$ is a field and G is an abelian group. Then,*

$$\mathbb{K}G = \{\sum_{i=1}^{n} \alpha_i g_i : \alpha \in \mathbb{K}, g \in G\}$$

where

$$\begin{aligned} \Delta : \mathbb{K}G \to \mathbb{K}G \bigotimes \mathbb{K}G \ \textit{by} \ \Delta(g) \mapsto (g \otimes g). \\ \textit{and} \ \epsilon : \mathbb{K}G \to \mathbb{K} \quad \textit{by} \quad \epsilon(g) \mapsto 1_{\mathbb{K}}. \end{aligned}$$

is a coalgebra over the field $\mathbb{K}$.

Proof. To prove $\mathbb{K}G$ is a coalgebra, we must show Δ is coassociative and $\mathbb{K}G$ has a multiplicative identity, a counit. Consider,

$$\begin{aligned} \Delta(1_{\mathbb{K}}1_G g) &= (1_{\mathbb{K}}1_G g \otimes 1_{\mathbb{K}}1_G g) \\ &= (1_{\mathbb{K}}1_G)(g \otimes 1_{\mathbb{K}}1_G g) \\ &= (1_{\mathbb{K}}1_G)(1_{\mathbb{K}}1_G)(g \otimes g) \\ &= (1_{\mathbb{K}}1_{\mathbb{K}}1_G 1_G)(g \otimes g) \\ &= 1_{\mathbb{K}}1_G(g \otimes g) \\ &= (1_{\mathbb{K}} g \otimes 1_G g) \\ &= (g \otimes g). \end{aligned}$$

Similarly,

$$\begin{aligned} \Delta(g 1_{\mathbb{K}}1_G) &= (g1_{\mathbb{K}}1_G \otimes g1_{\mathbb{K}}1_G) \\ &= (1_{\mathbb{K}}1_G)(g \otimes g1_{\mathbb{K}}1_G) \\ &= (1_{\mathbb{K}}1_G)(1_{\mathbb{K}}1_G)(g \otimes g) \\ &= (1_{\mathbb{K}}1_{\mathbb{K}}1_G 1_G)(g \otimes g) \\ &= 1_{\mathbb{K}}1_G(g \otimes g) \\ &= (g1_{\mathbb{K}} \otimes g1_G) \\ &= (g \otimes g). \end{aligned}$$

Therefore, $1_{\mathbb{K}}1_G \in \mathbb{K}G$ is the multiplicative identity in $\mathbb{K}G$.

To show Δ is coassociative, take an element and push it through the diagram. If Δ is coassociative, the diagram commutes and the results will be equal.
Let $c \in C$. Consider,
$\Delta(c) = c \otimes c$
$(1 \otimes \Delta)(c \otimes c) = c \otimes (c \otimes c)$
Going the other direction,
$\Delta(c) = c \otimes c$
$(1 \otimes \Delta)(c \otimes c) = (c \otimes c) \otimes c.$
Since $c \otimes (c \otimes c) = c \otimes c \otimes c = (c \otimes c) \otimes c$, Δ is coassociative. $\square$

4.3 Bialgebras

Definition 4.8. Let A be a $\mathbb{K}$-vector space, (A, m, u) an algebra and (A, Δ, ϵ) a coalgebra. Then $(H, m, u, \Delta, \epsilon)$ is a bialgebra and the following equivalent statements hold:

1. m and u are coalgebra maps.
2. Δ and ϵ are algebra maps.
3. The following diagrams commute:

$$\begin{tikzcd}
& A\otimes A \arrow[ld, "\Delta\otimes\Delta"'] \arrow[rd, "m"] & \\
A\otimes A\otimes A\otimes A \arrow[d, "1\otimes T\otimes 1"'] & & A \arrow[ldd, "\Delta"] \\
A\otimes A\otimes A\otimes A \arrow[rd, "m\otimes m"'] & & \\
& A\otimes A &
\end{tikzcd}$$

$$\begin{tikzcd}
A\otimes A \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "m"'] & \mathbf{k}\otimes\mathbf{k} \arrow[d, "m"] \\
A \arrow[r, "\epsilon"'] & \mathbf{k}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{k} \arrow[r, "u"] \arrow[d, "\Delta"'] & A \arrow[d, "\Delta"] \\
\mathbf{k}\otimes\mathbf{k} \arrow[r, "u\otimes u"'] & A\otimes A
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{k} \arrow[rr, "1"] \arrow[rd, "u"'] & & \mathbf{k} \\
& A \arrow[ru, "\epsilon"'] &
\end{tikzcd}$$

Figure 4: Bialgebras

Theorem 4.9. *$\mathbb{K}G$ is a bialgebra.*

Proof. The proof of this uses the commutativity of the previous diagrams, which can be found in [4] and [5]. [5] provides a pictoral proof of the commutativity of the pentagonal diagram. Note: the map T is the twist map defined as $T : B \bigotimes A \to A \bigotimes B$ by $T(b \otimes a) \mapsto (a \otimes b)$.

$$\begin{tikzcd}
& t_g\otimes t_h \arrow[ld, "\Delta\otimes\Delta"'] \arrow[rd, "m"] & \\
t_g\otimes t_g\otimes t_h\otimes t_h \arrow[d, "1\otimes T\otimes 1"'] & & t_{gh} \arrow[ldd, "\Delta"] \\
t_g\otimes t_h\otimes t_g\otimes t_h \arrow[rd, "m\otimes m"'] & & \\
& t_{gh}\otimes t_{gh} &
\end{tikzcd}$$

Figure 5: Commutativity elementwise

The rest of the proof is left to the reader. $\square$

4.4 Homology Group

Definition 4.10. Let (A, M_A, u_A) and (B, M_B, u_B) be two $\mathbb{K}$-algebras. The map $f : A \to B$ is a morphism of algebras if the following diagrams commute:

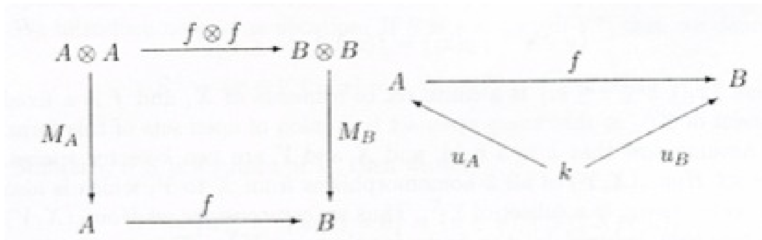

Figure 6: Morphism of Algebras

Additionally, let $(C, \Delta_C, \epsilon_c)$ and $(D, \Delta_D, \epsilon_D)$ be two $\mathbb{K}$-coalgebras. The map $g : C \to D$ is a morphism of coalgebras if the following diagrams commute:

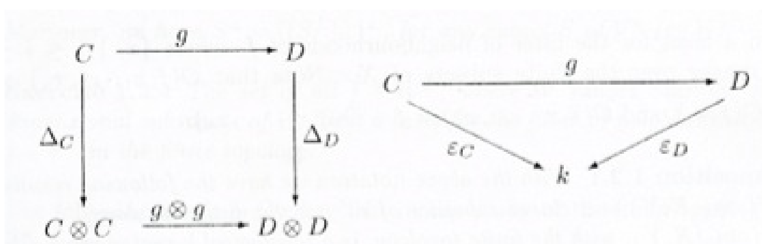

Figure 7: Morphism of Coalgebras

Furthermore, let H and L be two $\mathbb{K}$-bialgebras. The map $f : H \to L$ is a morphism of bialgebras if it is a morphism of algebras and a morphism of coalgebras between the underlying algebras.

Proof. Using previous proofs as a model, this proof is left to the reader. It can also be found in [4]. $\square$

Using morphisms to map between algebras, coalgebras, and bialgebras is a convenient tool and an important aspect of Hopf algebra and its applications. Additionally, the morphisms between an algebra and a coalgebra are equally intriguing. Furthermore, the set of all of these morphisms form a group called a homology group and is explored in the following section.

5 Hopf Algebra

Definition 5.1. Given C a coalgebra and A an algebra, the homology group of C and A, $Hom(C, A)$, is the set of all homomorphisms from C to A.

Theorem 5.2. *If $Hom(C, A)$ is the homology group of C and A and has a multiplication, $*$, defined as:*

$$(f * g)(c) = \sum f(c_1)g(c_2)$$

for any $f, g \in Hom(C, A)$ and for any $c \in C$,

then $Hom(C, A)$ is an algebra.

Proof. [4] Consider,

$$(f * (u\epsilon))(c) = \sum f(c_1)(u\epsilon)(c_2) = \sum f(c_1)\epsilon(c_2)1 = f(c)$$

Hence, $f * (u\epsilon) = f$. Similarly, $(u\epsilon) * f = f$. So, $u\epsilon$ is the identity element.

Furthermore, $*$ is associative. For $f, g, h \in Hom(C, A)$ and $c \in C$,

$$\begin{aligned}((f * g) * h)(c) &= \sum (f * g)(c_1)h(c_2) \\ &= \sum f(c_1)g(c_2)h(c_3) \\ &= \sum f(c_1)(g * h)(c_2) \\ &= (f * (g * h))(c)\end{aligned}$$

Therefore, $Hom(C, A)$ with $*$ as its associative multiplication and $u\epsilon$ as its identity is an algebra. $\square$

Lemma 5.3. Let H be a bialgebra. Denote the underlying coalgebra as H^c and H^a as the underlying algebra. $Hom(H^c, H^a)$ is an algebra with $*$ defined as:

$$(f * g)(h) = \sum f(h_1)g(h_2)$$

for any $f, g \in Hom(H^c, H^a)$ and $h \in H$ and the identity element is uϵ.
Note that the identity map, $I : H \to H$ is an element of $Hom(H^c, H^a)$.

Definition 5.4. Let H be a bialgebra. A linear map $S : H \to H$ is called an antipode of the bialgebra H if S is the inverse of the identity map $I : H \to H$ with respect to $*$ in $Hom(H^c, H^a)$.

Proposition 5.5. In a Hopf algebra, the antipode is unique and is the inverse of I in $Hom(H^c, H^a)$. $S : H \to H$ is written as $S * I = I * S = u\epsilon$ where $u\epsilon$ is the identity element in $Hom(H^c, H^a)$. In Sweedler notation:

$$\sum S(h_1)h_2 = \sum h_1 S(h_2) = \epsilon(h)1.$$

We have now finally built up enough structure to introduce and define a Hopf algebra.

Definition 5.6. A bialgebra A having an antipode S is called a Hopf algebra, and the following diagram commutes:

$$\begin{tikzcd}
& A \otimes A \arrow[rr, "S\otimes 1_A"] & & A \otimes A \arrow[dr, "m"] & \\
A \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & \mathbf{k} \arrow[rr, "u"] & & A \\
& A \otimes A \arrow[rr, "1_A\otimes S"'] & & A \otimes A \arrow[ur, "m"'] &
\end{tikzcd}$$

Figure 8: Hopf Algebra

Theorem 5.7. $\mathbb{K}G$ *is a Hopf algebra.*

Proof. We have already shown $\mathbb{K}G$ is a bialgebra. Let us define an antipode, S.

$$S : \mathbb{K}G \to \mathbb{K}G \text{ where } S(g) \mapsto g^{-1} \text{ for any } g \in G \text{ and extend linearly.}$$

Since $\mathbb{K}$ is a field and G is a group, every element in $\mathbb{K}$ and in G has an inverse, which makes g^{-1} valid. Additionally, S satisfies Proposition 5.5 since

$$\sum (g_1)g_2 = S(g)g = g^{-1}g = 1 = \epsilon(g)1.$$

Therefore, S is the antipode of bialgebra $\mathbb{K}G$, and $\mathbb{K}G$ is a Hopf algebra. $\square$

Provided is just a basic definition and a basic example of a Hopf algebra. There are many interesting theorems, additional examples, and important applications of Hopf algebras. However, these components are beyond the scope of this paper.

6 Conclusion

Even though mastering Hopf algebras was not a goal of this paper, it is expected that the reader has gained an understanding and appreciation of Hopf algebras that has hopfully sparked curiosity for further study. Hopf algebras represent a diverse and enriching field of mathematics. These structures have a wide variety of applications but are also worthy of study in their own right. The references listed below are a good place to begin further research. [4] is especially useful and provides many more properties and uses of Hopf algebras. Also, [1] is a collection of lecture videos and notes from a course on Hopf algebras and combinatorics and is an engaging and clear way to delve into applications. A good understanding of Hopf algebras opens many avenues of further study in many branches of mathematics. The purpose of this paper was to provide a basis for that understanding and incite curiosity to discover these applications.

7 References

[1] Ardila, Federico. *Hopf algebras and combinatorics.* Course notes, San Francisco State University. (2012).

[2] Cartier, Pierre. *A primer of Hopf algebras.* In Frontiers in Number Theory, Physics, and Geometry II, pp. 537-615. Springer Berlin Heidelberg, (2007).

[3] Conrad, Keith. *Tensor Products.* Course notes, Middle East Technical University.

[4] Dascalescu, Sorin, Constantin Nastasescu, and Serban Raianu. *Hopf algebra: An introduction.* (2000).

[5] Grinberg, Darij and Victor Reiner. *Hopf algebras in combinatorics.*(2013).

[6] Judson, Thomas W. *Abstract Algebra.* (2010).

[7] Wolfram Mathworld. *Various definitions.*