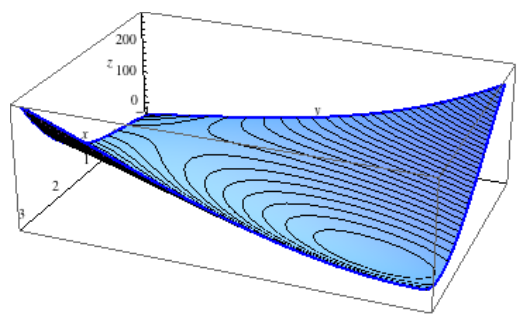
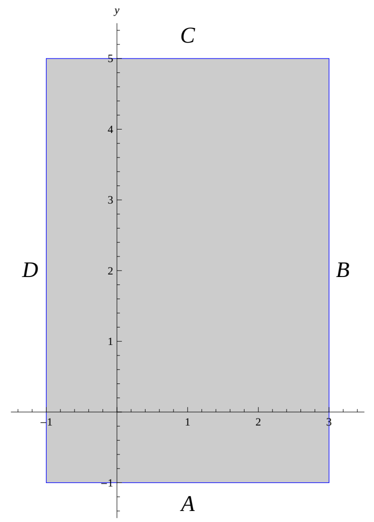
A global extremes example

Problem: Find the global extremes of the function $f(x,y) = 8x^3 - 24xy + y^3$ for the region having $-1 \leq x \leq 3$ and $-1 \leq y \leq 5$.





Solution: Our solution will consist of two main steps:

1. Find all critical points inside the region and compute the output of the function for each critical point.
2. Analyze the function along the boundary of the region. In this case, the boundary consists of the four edges of the rectangle as shown in the figure above so we will break this step into four smaller steps.

We now carry out the details of each step.

1. To find the critical points, we compute the partial derivatives

   $$f_x(x,y) = 8x^2 - 24y \qquad \text{and} \qquad f_y(x,y) = -24x + 3y^2.$$

   We set these equal to zero and solve to find the two critical points $(0,0)$ and $(2,4)$. For these, we compute $f(0,0) = 0$ and $f(2,4) = -64$. These two outputs are on our list of potential global extremes.

2. We next analyze the function along each of the four pieces of the boundary as denoted in the figure above.

   (A) This edge is defined by $y = -1$ so the function restricted to this edge is $a(x) = f(x,-1) = 8x^3 + 24x - 1$ for $-1 \leq x \leq 3$. For this restricted function, we compute

   $$a'(x) = 24x^2 + 24$$

   and set this equal to zero. This resulting equation has no solutions so this restricted function has no critical points. So, we need only compute the outputs for the two endpoints: $a(-1) = f(-1,-1) = -33$ and $a(3) = f(3,-1) = 287$.

(B) The edge B is defined by $x = 3$ so the function restricted to this edge is $b(y) = f(3, y) = 216 - 72y + y^3$ for $-1 \le y \le 5$. The derivative is

$$b'(y) = -72 + 3y^2.$$

Setting this equal to zero and solving gives the critical point $y = \sqrt{24}$. So, we compute $b(\sqrt{24}) = f(3, \sqrt{24}) = 216 - 48\sqrt{24} \approx -19.15$. The outputs for the endpoints here are $b(-1) = f(3, -1) = 287$ and $b(5) = f(3, 5) = -19$. Note that we already calculated $f(3, -1) = 287$ in looking at the right end of edge A.

(C) For edge C, we have $y = 5$ and the restricted function $c(x) = f(x, 5) = 8x^3 - 120x + 125$ for $-1 \le x \le 3$. The derivative is

$$c'(x) = 24x^2 - 120.$$

Solving the equation given by setting this equal to zero, we get the critical point $x = \sqrt{5}$. So, the outputs we need to compute are $c(\sqrt{5}) = f(\sqrt{5}, 5) = -80\sqrt{5} + 125 \approx -53.88$, $c(-1) = f(-1, 5) = 237$, and $c(3) = f(3, 5) = -19$. Again, note that the right endpoint for edge C is the top endpoint for edge B so we have previously put $f(3, 5) = -19$ on the list.

(D) For the last edge, we have $x = -1$ and the restricted function $d(x) = f(-1, y) = -8 + 24y + y^3$. Computing the derivative and setting it equal to zero gives an equation that has no solutions so there are no critical points for this restricted function. We have already computed the outputs for the two endpoints since the top endpoint of D is the left endpoint of C and the bottom endpoint of D is the left endpoint of A.

Our results are gathered in the table below.

Input (x, y)	Output $f(x, y)$	Note
$(0, 0)$	0	critical point for f
$(2, 4)$	-64	critical point for f
$(-1, -1)$	-33	endpoint of edges A and D
$(3, -1)$	287	endpoint of edge A and B
$(3, \sqrt{24}$	$216 - 48\sqrt{24} \approx -19.15$	critical point for restricted function along edge B
$(3, 5)$	-19	endpoint of edge B and C
$(\sqrt{5}, 5$	$-80\sqrt{5} + 125 \approx -53.88$	critical point for restricted function along edge C
$(-1, 5)$	273	endpoint of edge C and D

To finish, we just need to read off the smallest and largest outputs. From the table, we see that the global minimum is $f(2, 4) = -64$ and the global maximum is $f(3, -1) = 287$.