Mathematics 434 Spring 2011 Problem Sheet 10

Due April 18

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Name

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. **Only write on one side of each page.**

"The road to wisdom? Well it plain and simple to express: Err and err and err again, but less and less and less." -Piet Hein, poet and scientist (1905-1996)

Problems

1. Prove that if two polynomials f, g with coefficients in a field F factor into linear factors in F, then their greatest common divisor is the product of their common linear factors.

2. Factor the following polynomials into irreducible factors in $\mathbf{F}_p[x] = \mathbf{Z}_p[x]$
   (a) $x^3 + x + 1$, $p = 2$
   (b) $x^2 - 3x - 3$, $p = 5$
   (c) $x^2 + 1$, $p = 7$

3. Adapt Euclid's proof of the infinitude of prime integers to show that for any field F, there are infinitely many monic irreducible polynomials in $F[x]$.
   (a) Also explain why this argument fails for the formal power series ring $F[[x]]$.

4. Chinese Remainder Theorem
   (a) Let n, m be relatively prime integers and let a, b be arbitrary integers. Prove that there is an integer x which solves the simultaneous congruence $x \equiv a \pmod{m}$ and $x \equiv b \pmod{n}$.
   (b) Determine all solutions of this system of congruences.

5. Solve the following congruences
   (a) $x \equiv 3 \pmod{15}$, $x \equiv 5 \pmod{8}$, $x \equiv 2 \pmod{7}$
   (b) $x \equiv 13 \pmod{43}$, $x \equiv 7 \pmod{71}$.

6. Partial Fractions for polynomials
   (a) Prove that every rational function in $\mathbf{C}[x]$ can be written as a sum of a polynomial and a linear combination of functions of the form $1/(x-a)^i$.
   (b) Find a basis for $\mathbf{C}(x)$ as a vector space over $\mathbf{C}$.

7. Let a and b be relatively prime integers. Prove there are integers m, n such that $a^m + b^n \equiv 1 \pmod{ab}$