# Mathematics 434 Spring 2011 Problem Sheet 9

**Due April** 11

**Name**

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. **Only write on one side of each page.**

"*Every now and then go away, have a little relaxation, for when you come back to your work your judgment will be surer. Go some distance away because then the work appears smaller and more of it can be taken in at a glance and a lack of harmony and proportion is more readily seen.*" – Leonardo Da Vinci

## Problems

1. Let $a, b$ be positive integers whose sum is a prime $p$. Prove their greatest common divisor is 1.
2. Define the greatest common divisor of a set of $n$ integers.
   (a) Prove its existence.
   (b) Prove that if $d$ is the greatest common divisor of the set of integers $\{a_1, \cdots, a_n\}$, then the greatest common divisor of $\{\frac{a_1}{d}, \cdots, \frac{a_n}{d}\}$ is 1.
3. Do both of the following
   (a) Let $a, b$ be integers with $a \neq 0$ and write $b = aq + r$ with $0 \leq r < |a|$ .Prove the greatest common divisors $\gcd(a, b)$ and $\gcd(a, r)$ are equal.
   (b) Describe an algorithm based on part $(a)$ for computing the greatest common divisor.
   (c) Use your algorithm to compute the greatest common divisors of the following:
      i. 1456, 235
      ii. 123456789, 135792468
4. Compute the greatest common divisor in $\mathbf{C}[x]$ of the two polynomials: $x^3 - 6x^2 + x + 4$ and $x^5 - 6x + 1$.
5. Prove that in a principal ideal domain, an irreducible element is prime.
6. Factor the following polynomials into irreducible factors in $(\mathbf{Z}/p\mathbf{Z})[x]$
   (a) $x^3 + x + 1$, $p = 2$
   (b) $x^2 - 3x - 3$, $p = 5$
   (c) $x^2 + 1$, $p = 7$
7. Do both of the following
   (a) Let $m, n$ be two integers. Prove their greatest common divisor in $\mathbf{Z}$ is the same as their greatest common divisor in $\mathbf{Z}[i]$.
   (b) Find the greatest common divisor of the two numbers $11 + 7i$ and $18 - i$ in $\mathbf{Z}[i]$.
8. Let $a, b$ be elements of a field $F$ with $a \neq 0$. Prove a polynomial $f(x) \in F[x]$ is irreducible if and only if $f(ax + b)$ is irreducible.

9. Prove the kernel of the evaluation homomorphism $\phi : \mathbf{Z}[x] \to \mathbf{R}$ with $\phi(f(x)) = f\left(1+\sqrt{2}\right)$ is a principal ideal and find a generator for this ideal.

10. Prove Gauss's Lemma without using reduction modulo $p$ in the following way: Let $a_i$ be the coefficient of lowest degree $i$ of $f$ which is not divisible by $p$. So $p$ divides $a_r$ if $r < i$ but $p$ does not divide $a_i$.Similarly let $b_j$ be the coefficient of lowest degree of $g$ which is not divisible by $p$. Prove the coeffiecient of $h$ of degree $i+j$ is not divisible by $p$.

11. Prove the following polynomials are irreducible in $\mathbf{Q}[x]$

    - $x^2 + 27x + 213$, $8x^3 - 6x + 1$, $x^5 - 3x^4 + 3$

12. Factor $x^5 + 5x + 5$ into irreducible factors in $\mathbf{Q}[x]$ and in $(\mathbf{Z}/2\mathbf{Z})[x]$.

13. Using reduction modulo 2 as an aid, factor the following polynomials in $\mathbf{Q}[x]$

    (a) $x^2 + 2345x + 125$
    (b) $x^4 + 2x^3 + 2x^2 + 2x + 2$
    (c) $x^4 + 2x^3 + 3x^2 + 2x + 2$
    (d) $x^5 + x^4 - 4x^3 + 2x^2 + 4x + 1$

14. Factor the following into primes in $\mathbf{Z}[i]$.

    - $30$, $1 - 3i$, $10$, $6 + 9i$