Geometry Overview and Outline

Flugs

- Axiomatic systems by example: Scorpling Flugs
- Introduced the formalist position that
  1. undefined terms are 'meaningless'
  2. meaning is obtained through models (examples of the system)
  3. Systems are abstract: models are specific (but are parts of other abstract logical systems: e.g., hyperbolic model in terms of Euclidean geometry.)

Logic

- Basics of the predicate calculus: an axiomatic system (not in depth)
- Truth tables of basic predicates
- Quantifiers are essential.
- Justifications for methods of proof
  1. Direct $(H \wedge (H \Rightarrow C)) \Rightarrow C$ (Modus Ponens)
  2. Contrapositive: $(H \Rightarrow C) \Longleftrightarrow (\tilde{}C \Rightarrow \tilde{}H)$
  3. Contradiction: $((H \wedge \tilde{}C) \Rightarrow (D \wedge \tilde{}D)) \Rightarrow C$

Proof Techniques:

- Forward-Backward analysis
- Add structure:
  1. Proof by cases
  2. Focus on a local situation
  3. "Find something that works"
  4. Proof by contradiction
- Remove structure (generalize)
  1. Remove a hypothesis
  2. Refuse to use a particular axiom

Geometry

The Buildup

- **Incidence geometry** and what is deducible therefrom

    1. Introduction to models and their value with respect to unprovability and independence.
        (a) Any statement that "does not make sense" in a model, cannot be deduced from the axioms.
        (b) The exact meaning of "does not make sense" in a model is that the interpreted statement cannot be proven in the axiomatic system in which the model is interpreted. *Example: In Chapter 7 we have the Klein interpretation of hyperbolic geometry in terms of Euclidean objects. Thus, a hyperbolic statement S "makes sense" in the Klein interpretation if and only if we can use the Euclidean axioms to prove the Euclidean statement T that is the Klein interpretation of S.*
    2. Finite geometries
    3. Projective planes
        (a) Order, number of points, number of lines
    4. Projective completions of affine planes.

- **Betweenness and Incidence geometry**

- **Congruence, Betweenness and Incidence geometry**

- **Continuity, Congruence, Betweenness and Incidence geometry**

    1. Sophistication of Dedekind's axiom. We used only once (in chapter 7)

Neutral geometry

- The common structure of both Euclidean and hyperbolic geometry

- Equivalents of Euclid V

    1. We know at least 10
    2. These define a 'conceptual boundary' distinguishing what can be proven in Euclidean geometry from what can be proven in hyperbolic geometry.
    3. Specifically, the only statements in neutral geometry that are equivalent to Hilbert's parallel property are the ones that are theorems of Euclidean geometry and whose negations are theorems of hyperbolic geometry.

Hyperbolic geometry

- What new (and possibly counter-intuitive) results can be deduced if we add the negation of Hilbert to the axioms of neutral geometry.

- Our fundamental tool was that rectangles do not exist.

Meta Mathematics

- Theorem: If Euclidean geometry is consistent then so is hyperbolic geometry.
  - Corollary: If you can prove that Euclid V (or its negation) follows from the axioms of neutral geometry, then Euclidean geometry is inconsistent.
- Method of proof of the Meta Mathematical Theorem
  1. Construct a **model** of hyperbolic geometry inside Euclidean geometry.
  2. Expect to have to explain
     (a) What it means for a model to be "inside" Euclidean geometry
     (b) The details of why the existence of such a model proves the Meta theorem.
- The actual proof
  1. (Congruence Axiom 6 involves much study of inversion in Euclidean circles which we did not cover in class.)
  2. Defined the Klein and Poincaré disk interpretations
  3. Showed some hyperbolic axioms held in Klein
  4. Exhibited an isomorphism (neglecting congruence) between Klein and Poincaré that preserved points, lines, incidence, and betweenness.
  5. Showed the congruence axioms held in Poincaré disk and **defined** the interpretation of congruence in the Klein disk so that the isomorphism preserved congruence as well.
  6. Thus, both Klein and Poincaré disks are models of hyperbolic geometry in Euclidean geometry.
- Inversion in Circles facts

1. γ, δ circles with O not on δ, then
   (a) δ' is also a circle (inverse with respect to γ)
   (b) $\delta \perp \gamma$ iff $\delta = \delta'$ (inverse with respect to γ)
   (c) Lines not through O invert to punctured circles
   (d) Punctured circles invert to lines
   (e) If $\delta \perp \gamma$ and we **invert through** δ then
      i. $\gamma' = \gamma$
      ii. P on γ iff P' on γ
      iii. P interior to γ iff P' interior to γ
      iv. $d(AB) = d(A'B')$
      v. $(\measuredangle ABC)^\circ = (\measuredangle A'B'C')^\circ$
      vi. For each P interior to γ there is a circle δ where
         A. $\sigma \perp \gamma$
         B. $P' = O$ (inversion with respect to δ)