Chapter 1

The Basic Constructs

1.A Topological Spaces and Continuity

Problem 1.A.1

Problem 1.A.2 *Find all possible topologies for the set* $X = \{a, b, c\}$

Proof 1) $\{\emptyset, \{a, b, c\}\}$
2) $\{\emptyset, \{a, b, c\}, \{a\}\}$
3) $\{\emptyset, \{a, b, c\}, \{b\}\}$
4) $\{\emptyset, \{a, b, c\}, \{c\}\}$
5) $\{\emptyset, \{a, b, c\}, \{a, b\}\}$
6) $\{\emptyset, \{a, b, c\}, \{a, c\}\}$
7) $\{\emptyset, \{a, b, c\}, \{b, a\}\}$
8) $\{\emptyset, \{a, b, c\}, \{a, b\}, \{a\}\}$
9) $\{\emptyset, \{a, b, c\}, \{a, b\}, \{b\}\}$
10) $\{\emptyset, \{a, b, c\}, \{a, b\}, \{c\}\}$
11) $\{\emptyset, \{a, b, c\}, \{a, c\}, \{a\}\}$
12) $\{\emptyset, \{a, b, c\}, \{a, c\}, \{b\}\}$
13) $\{\emptyset, \{a, b, c\}, \{a, c\}, \{c\}\}$
14) $\{\emptyset, \{a, b, c\}, \{b, c\}, \{a\}\}$
15) $\{\emptyset, \{a, b, c\}, \{b, c\}, \{b\}\}$
16) $\{\emptyset, \{a, b, c\}, \{b, c\}, \{c\}\}$
17) $\{\emptyset, \{a, b, c\}, \{a, b\}, \{a, c\}, \{a\}\}$
18) $\{\emptyset, \{a, b, c\}, \{a, b\}, \{b, c\}, \{b\}\}$
19) $\{\emptyset, \{a, b, c\}, \{a, c\}, \{b, c\}, \{c\}\}$
20) $\{\emptyset, \{a, b, c\}, \{a, b\}, \{a\}, \{b\}\}$
21) $\{\emptyset, \{a, b, c\}, \{a, c\}, \{a\}, \{c\}\}$
22) $\{\emptyset, \{a, b, c\}, \{b, c\}, \{b\}, \{c\}\}$
23) $\{\emptyset, \{a, b, c\}, \{a, b\}, \{a, c\}, \{a\}, \{b\}\}$
24) $\{\emptyset, \{a, b, c\}, \{a, b\}, \{b, c\}, \{a\}, \{b\}\}$
25) $\{\emptyset, \{a, b, c\}, \{a, c\}, \{b, c\}, \{a\}, \{c\}\}$
26) $\{\emptyset, \{a, b, c\}, \{a, b\}, \{a, c\}, \{a\}, \{c\}\}$
27) $\{\emptyset, \{a, b, c\}, \{a, b\}, \{b, c\}, \{b\}, \{c\}\}$
28) $\{\emptyset, \{a, b, c\}, \{a, c\}, \{b, c\}, \{b\}, \{c\}\}$
29) $\{\emptyset, \{a, b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\}\}$ ■

Problem 1.A.3 *True of False? If $(X,\mathcal{U})$ is a topological space and $\mathcal{C}$ is a family of open sets in X, then $\bigcap\{C : C \in \mathcal{C}\}$ is open.*

Proof False. Let $X = \mathbb{R}^1$, let $\mathcal{U}$ be the usual topology on $\mathbb{R}^1$ and let $\mathcal{C} = \{(-1/n, 1/n) : n \in \mathbb{N}\}$. We claim $\{0\} = \bigcap\{C : C \in \mathcal{C}\}$, which is not open.
Clearly $0 \in \bigcap\{C : C \in \mathcal{C}\}$. Now we show 0 is the only element of $\bigcap\{C : C \in \mathcal{C}\}$. Without loss of generality choose an arbitrary x such that $0 < x < 1$. Choosing $N \in \mathbb{N}$ sufficiently large, we can find a number $1/N$ such that $x > 1/N$. Then $x \notin (-1/N, 1/N) \in \mathcal{C}$, and therefore $x \notin \bigcap\{C : C \in \mathcal{C}\}$. A similar proof holds if $-1 < x < 0$ and therefore 0 is the only element of $\bigcap\{C : C \in \mathcal{C}\}$, and thus $\bigcap\{C : C \in \mathcal{C}\}=\{0\}$ is not open. ■

Problem 1.A.4 *For each subset A of a nonempty set X, Let $\mathcal{F}(A)$ be the topology on X whose open sets are $\{\}, X$ and all subsets of X containing A. Assume that X has at least two elements.*

1. *Show that $A \subset B$ if and only if $\mathcal{F}(B) \subset \mathcal{F}(A)$.*
2. *Suppose that $A_1, A_2, \cdots, A_n$ are subsets of X and that $\mathcal{F}$ is a topology for X such that $\mathcal{F}(A_i) \subset \mathcal{F}$ for each i. Show that $\mathcal{F}(\cap_{i=1}^n A_i) \subset \mathcal{F}$.*

Proof We first note that any union of sets in X that contain A will still contain A as will any finite intersection of sets that contain A. Hence $\mathcal{F}(A)$ really is a topology on the set X.
To show $A \subset B$ if and only if $\mathcal{F}(B) \subset \mathcal{F}(A)$ first suppose $A \subset B$ and let $O \in \mathcal{F}(B)$ be an open set in that topology. By the definition of $\mathcal{F}(B)$ we know that $B \subset O$ and since $A \subset B$ then we have $A \subset B \subset O$ and O satisfies the defining condition to be in the topology $\mathcal{F}(A)$. On the other hand, if $\mathcal{F}(B) \subset \mathcal{F}(A)$ then any set U that is open in the topology $\mathcal{F}(B)$ is also open in the topology $\mathcal{F}(A)$. In particular, the set B is open in $\mathcal{F}(B)$ so it is also open in $\mathcal{F}(A)$ and by the definition of the latter topology $A \subset B$.

For the second part of the theorem, consider the special case when $n = 2$ and let U be an open set in $\mathcal{F}(A_1 \cap A_2)$. Thus, we know $(A_1 \cap A_2) \subset U$. We also know that if we enlarge U we can obtain sets that are open in $\mathcal{F}(A_1)$ and $\mathcal{F}(A_2)$. Specifically, since $A_1 \subset (A_1 \cup U)$ the first part of this problem tells us that $(A_1 \cup U) \in \mathcal{F}(A_1)$. That is, $(A_1 \cup U)$ is an open set in the topology $\mathcal{F}(A_1)$. Since $\mathcal{F}(A_1) \subset \mathcal{F}(A)$ we deduce $(A_1 \cup U)$ is also open in $\mathcal{F}$. In a similar fashion we have that $(A_2 \cup U)$ is open in $\mathcal{F}$. Since $\mathcal{F}$ is a topology, the intersection $(A_1 \cup U) \cap (A_2 \cup U) = (A_1 \cap A_2) \cup U = U$ is open in $\mathcal{F}$. [Note that we used $(A_1 \cap A_2 \subset U)$ in the last step.]
The general case is essentially the same but with a bit more notational bookkeeping. Specifically, If $\cap_{i=1}^n A_i \subset U$, then $A_i \cup U$ is open in $\mathcal{F}(A_i) \subset \mathcal{F}$ and so the finite intersection of sets that are open in $\mathcal{F}$ given by $\cap_{i=1}^n (A_i \cup U) = \cap_{i=1}^n A_i \cup U = U$ is open in $\mathcal{F}$ as we wish to show.
■

Problem 1.A.5 *Find 4 equivalent bases for the usual topology on $\Re^2$*
For each basis, let $B_\epsilon = \left\{S_\epsilon^d(x) : x \in X, \epsilon > 0\right\}$

Proof

1. $S_\epsilon^{d_1}(x) = \left\{y \in X : \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} < \epsilon\right\}$ (Circles)
2. $S_\epsilon^{d_2}(x) = \left\{y \in X : \sqrt{\alpha(x_1 - y_1)^2 + \beta(x_2 - y_2)^2} < \epsilon, with \alpha, \beta \in \Re^+\right\}$ (Elipses)
3. $S_\epsilon^{d_3}(x) = \{y \in X : |x_1 - y_1| + |x_2 - y_2| < \epsilon\}$ (Diamonds)

4. $S_\epsilon^{d_4}(x) = \{y \in X : |x_1 - y_1| < \epsilon, |x_2 - y_2| < \epsilon\}$ (Squares)

I will show, as an example, that the square and diamond bases are equivalent.

Given a diamond basis $S_\epsilon^{d_3}(x)$.
There is a square basis $S_{\frac{\epsilon}{3}}^{d_4}(x)$.

Each point y in the square basis must satisfy $|x_1 - y_1| < \frac{\epsilon}{3} and |x_2 - y_2| < \frac{\epsilon}{3}$.
Therefore, $|x_1 - y_1| + |x_2 - y_2| < \frac{2\epsilon}{3} < \epsilon$.
Therefore, each y in the square basis satisfies the requirements to be in the diamond basis.

Given a square basis $S_\epsilon^{d_4}(x)$.
There is a diamond basis $S_{\frac{\epsilon}{3}}^{d_3}(x)$.

Each point y in the diamond basis must satisfy $|x_1 - y_1| + |x_2 - y_2| < \frac{\epsilon}{3}$.

Since each absolute value is greater or equal to zero,
$0 \leq |x_1 - y_1| < \frac{\epsilon}{3} and 0 \leq |x_2 - y_2| < \frac{\epsilon}{3}$.
Therefore, $|x_1 - y_1| + |x_2 - y_2| < \frac{2\epsilon}{3} < \epsilon$.
Therefore, each y in the diamond basis satisfies the requirements to be in the square basis.

Therefore, the two bases are equivalent. ■

Problem 1.A.6

Problem 1.A.7 *Suppose $\Re^1$ has the usual topology. Show that a function $f : \Re^1 \to \Re^1$ is continuous if and only if $f^{-1}(-\infty, a)$ and $f^{-1}(a, \infty)$ are open for each $a \in \Re^1$.*

Proof
Let $(\Re^1, U)$ be the domain of f with the usual topology and $(\Re^1, V)$ be the co-domain such that $f(u) \subset V$ where $u \in U$.
" $\Rightarrow$ " Suppose that f is continuous. $\forall b \in f^{-1}(-\infty, a)$ such that $f(b) \subset (-\infty, a)$. $\exists u_b \in U$ such that $b \in u_b$. Since the function is continuous $f(u_b) \subset (-\infty, a)$ and $u_b \subset f^{-1}(-\infty, a)$. Therefore by definition 1.A.1 (i) $f^{-1}(-\infty, a)$ must be open since $f^{-1}(-\infty, a) = \bigcup\{u_b : b \in b \in f^{-1}(-\infty, a)\}$. The same argument holds for $f^{-1}(a, \infty)$.
" $\Leftarrow$ " Suppose that $f^{-1}(-\infty, a)$ and $f^{-1}(a, \infty)$ are open sets in U. By theorem 1.A.6 $(-\infty, a)$ and (a, ∞) are a basis for V since (i) $\Re^1 = (-\infty, a) \cup (a, \infty)$ and (ii) $(-\infty, a) \cap (a, \infty) = \phi$. Therefore by excercise 1.A.11 the function f is continuous since $f^{-1}(-\infty, a)$ and $f^{-1}(a, \infty)$ are open sets in U.
■

Problem 1.A.8 *Show that a subset A of $\mathbf{R}^1$ is open in the usual topology if and only if A can be written as the countable union of pairwise disjoint open intervals (intervals of the form $(-\infty, a)$, (b, ∞), $(-\infty, \infty)$ are allowed).*

Proof Suppose first that A can be written as the countable union of pairwise disjoint open intervals: $A = \cup_{i=1}^{\infty} O_i$. Since open intervals are open sets in the usual topology, then A equals the union of open sets and is hence must be open in that topology.

Now suppose that A is an open set in the usual topology on $\mathbb{R}^1$. For each $a \in A$ let
$O_a = \cup\{I_\alpha : \alpha \in \Lambda,\ I_\alpha \text{ is an open interval with } a \in I_\alpha \subset A\}$ .That is, O_a is the union of all open intervals contained in A that have a as an element. Our argument requires that we know O_a is an open interval contained in A and that if some $b \neq a$ satisfies $b \in O_a$ then $O_b = O_a$.
To show that O_a is an interval, first note that if $x \neq a$ is in O_a then x and a are both in a single interval I_α. Now suppose x, y are distinct numbers in O_a.Then by the definition of O_α there are two open intervals I_α and I_β both contained in A and where $a, x \in I_\alpha$ and $a, y \in I_\beta$. Since I_α and I_β are two intervals with non trivial intersection, their union is an interval and so contains every z between x and y. Since x and y were arbitrary in O_α this shows that O_α is an interval. Since O_a is also the union of open intervals contained in A then O_a is both open and $O_a \subset A$.
To show that the set $\{O_a : a \in A\}$ partitions X, suppose that $x \in O_a$ where $x \neq a$. Then $O_x \cup O_a$ is an interval containing both a and x so that $O_x \cup O_a \subset O_a$ and $O_x \cup O_a \subset O_b$. Since $O_a \subset O_x \cup O_a$ and $O_x \subset O_x \cup O_a$ we now have $O_a = O_x \cup O_a = O_x$ whenever $x \in O_a$. Thus we have proven that if $a, b \in A$ then either O_a and O_b are either the same set or they are disjoint.
Finally, since $A = \cup\{O_a : a \in A\}$ we know A is the union of disjoint open intervals. Since $\mathcal{B} = \{(a,b) : a, b \in \mathbf{Q}\}$ is a countable basis for the usual topology on $\mathbf{R}^1$ (the comment following definition 1.A.9 in the textbook) we know any open set can be written as the countable (because $\mathcal{B}$ is countable) union of basic open sets. Thus A is the countable union of basic open sets. Since each distinct O_a contains at least one basic open set, there can only be countably many distinct sets O_a. Thus any open set A in the usual topology on $\mathbf{R}^1$ can be written as a countable union of disjoint open intervals. ■

Problem 1.A.9 *Suppose (X,U) and (Y,V) are topological spaces. A function $f\colon X \to Y$ is open if and only if $f(\mathtt{U}) \in V$ whenever $\mathtt{U} \in U$. Find an open function that is not continuous.*

Proof **(1.A.4)Theorem.** Suppose (X,U) and (Y,V) are topological spaces and $f\colon X \to Y$. Then f is contnuous if and only if for each $\mathtt{V} \in V, f^{-1}(\mathtt{V}) \in U$.
$f\colon X \to Y$ where U is the indiscrete topology and V is the discrete topology. F is open but not continuous because $\exists$ v $\in V$ such that $f^{-1}(\mathtt{v}) \neg \in U$.
■

Problem 1.A.10 *Find an example of bases $\mathcal{B}$ and $\hat{\mathcal{B}}$ for topological spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$, and a continuous function $f : X \to Y$ such that there is a bases element $\hat{B} \in \hat{\mathcal{B}}$ for which $f^{-1}(\hat{\mathcal{B}}) \notin \mathcal{B}$*

Proof Let $X = Y = (0,\infty)$ and let $\mathcal{B} = \hat{\mathcal{B}}$=$\{(a,b) : 0 < a < b, a, b \in \mathbb{Q}\}$ be the bases for the topologies on X and Y. Define the function $f : (0,\infty) \to (0,\infty)$ by $f(x) = x/\pi$. Clearly f is continuous on $(0,\infty)$. We note $f^{-1} : (0,\infty) \to (0,\infty)$ is given by $f^{-1}(x) = \pi x$ since $f^{-1}(f(x)) = x\ \forall\, x \in (0,\infty)$. Now take $(a,b) = \hat{B} \in \hat{\mathcal{B}}$. Then $f^{-1}((a,b)) = (\pi a, \pi b)$. But since $a, b \in \mathbb{Q}$, we conclude $\pi a, \pi b \notin \mathbb{Q}$ since π is an irrational number and the product of a rational number and an irrational number is an irrational number. Therefore we conclude $(\pi a, \pi b) = f^{-1}((a,b)) = f^{-1}(\hat{B}) \notin \mathcal{B}$ and the example is complete. ■

1.B Further Examples of Topological Spaces

Problem 1.B.1

Problem 1.B.2

Problem 1.B.3 *Determine which subsets A of $\mathcal{E}^1$ have the property that both A and $\mathcal{E}^1 \backslash A$ are open.*

Proof The only subsets with the desired property are $\mathbb{R}$ and $\emptyset$. Clearly $\mathcal{E}^1\backslash\mathbb{R} = \emptyset$ which is open by the definition of a topological space. Also $\mathcal{E}^1\backslash\emptyset = \mathbb{R}$ which is also open by the definition of a toplogical space. Now we will show these are the only sets with this property. Let B be a basic open set of $\mathcal{E}^1$. Then B is of the form (a, b) with $a < b$. Then $\mathcal{E}^1\backslash B$ is of the form $(-\infty, a]\bigcup[b, \infty)$, which is not open by 1.A.8. In general, an open set C with $C \neq \mathbb{R}$ and $C \neq \emptyset$ is, by 1.A.8, a disjoint union of open intervals. Then, $\mathcal{E}^1\backslash C$ is the union of intervals of the form $(-\infty, a]\bigcup[b, c]\bigcup\ldots\bigcup[d, \infty)$, or $[a, b]\bigcup\ldots\bigcup[c, \infty)$, or $(-\infty, a]\bigcup\ldots\bigcup[b, c]$ which are not open.
■

Problem 1.B.4

Problem 1.B.5

Problem 1.B.6 *Show that $(X, \mathcal{U})$ is a topolgical space, and determine if it is T_2, first countable, or second countable where:*

- *X is an uncountable set and $\mathcal{U}$ is the countable complement topology;*
- *X is an uncountable set, p is a particular point in X, and $\mathcal{U}$ consists of $\emptyset$ together with those subsets A of X such that either $X \setminus A$ is countable or $p \notin A$;*
- *$X = [-1, 1]$, and $\mathcal{U}$ is the topology generated by a basis consisiting of sets of the form $[-1, b), (a, 1]$, and (a, b), where $a < 0$ and $0 < b$.*

Proof
Let X be an uncountable set and define U by taking the empty set together with all subsets A of X such that $X \setminus A$ is countable.
I. This is a topological space.
1. The null set is included in the topology and, as $X \setminus X$ is countable, so is the set X.
2. Any union of a subset A_1 with some other A_2 will yield another subset which has the property that $X \setminus A$ is countable. Unions with the null set return the set which was in the topology and unions with the set X return X, which are both in the topology.
3. Finite intersections of subsets of A will yield an associated $\bigcup X \setminus A_i$, which is the union of the countable complements and, since there are finitely many unions, this union of countable complements is countable. Therefore, the intersection of any subsets of A is in the topology. Intersections with the null set return the null and intersections with X return the original set, which are both in the topology.
II. This is NOT T_2.
$\forall x, y \in X$, $x \neq y$, any U_x, U_y for which $x \in U_x, y \in U_y$ will have the property that, since both of the sets are uncoutably infinite and that the complements of X with each set yields a countable set, $U_x \bigcap U_y \neq \{\}$.
III. This is NOT first countable.
For some subset A_x of A which contains some $x \in X$ and has uncountably many elements, there are uncountably many subsets of the subset A_x which contain x. Therefore, no countable neighborhood basis exists.
IV. This is NOT second countable.
Since this topological space is not first countable, it cannot be second countable. There is no countable neighborhood basis, so there cannot be a countable basis.

Part B. Let X be an uncountable set and let p be a point in X. Let U consist of the the null set together with the subsets A of X such that either $X \setminus A$ is countable or $p \notin A$.

I. This is a topological space.
1. The null set is included in the topology and, as $X \setminus X$ is countable, so is the set X.
2. Any union of a subset A_1 with some other A_2 will yield another subset which has the property that $X \setminus A$ is countable, if p is an element of both. Unions where p is not an element of either subset yields a subset which satisfies the condition that p is not in that set and is therefore in the topology. If p is an element of one, but not the other set to be unioned, the union will still have the property that the complement of X with the union will be countable. Unions with the null set return the set which was in the topology and unions with the set X return X, which are both in the topology.
3. Finite intersections of subsets of A will yield an associated $\bigcup X \setminus A_i$, which is the union of the countable complements when the subsets all contain p and, since there are finitely many unions, this union of countable complements is countable. Intersections where at least one subset does not contain p will yield a subset which satisfies the condition that p is in the complement of the intersection of those sets. Therefore, the intersection of any subsets of A is in the topology. Intersections with the null set return the null and intersections with X return the original set, which are both in the topology.
II. This is T_2.
$\forall x, y \in X$, $x \neq y$ and $x, y \neq p$, $\exists U_x, U_y$ for which $U_x = \{x\}, U_y = \{y\}$ and $U_x \bigcap U_y = \{\}$. If one of the points is p, then $\exists U_p = X \setminus \{x\}$, so that $U_x \bigcap U_p = \{\}$.
III. This is NOT first countable.
For some subset A_p of A which contains $p \in X$ and has uncountably many elements, there are uncountably many subsets of the subset A_p which contain p. Therefore, no countable neighborhood basis exists.
IV. This is NOT second countable.
Since this topological space is not first countable, it cannot be second countable. There is no countable neighborhood basis, so there cannot be a countable basis.

Part C. Let $X = [-1, 1]$ and let U be the topology formed from a basis containing sets of the form $[-1, b)$, $(a, 1]$ and (a, b), where $a < 0$ and $b > 0$. $a, b \in \Re$.
I. This is a topological space.
1. The null set is included in the topology and so is the set X.
2. By cases, the union of any set of the form $[-1, b_1)$ with a set of the form (a, b_2) will return a set of the form $[-1, max\{b_1, b_2\}$. Similarly, unions of sets of the form $(a_1, 1]$ with sets of the form (a_2, b) will return sets of the form $(max\{a_1, a_2, 1]\}$. Unions of (a, b) form sets with similar sets will return sets of the same form. Unions of $[-1, b)$ and $(a, 1]$ sets will return the whole set, $[-1, 1]$. All of these unions are contained within the topology, as they can be written as unions of basic open sets.
3. Finite intersections of sets which are not all of the form $[-1, b)$ or $(a, 1]$ will return a set of the form (a, b). Finite intersections of sets which are all of the same form will return sets of that same form. All of these intersections return sets which are in the topology, as they can be written as unions of basic open sets.
II. This is NOT T_2.
$\forall x, y \in X$, $x \neq y$ and $x < y$, observe the case where $x < y < 0$. Any set basic open sets of the form $[-1, b)$ must contain both points. Any set (a, b) or $(a, 1]$ of the other two forms in which $x \in (a, b) \subset (a, 1]$ will have the property that $a < x < y < b$ and therefore b is also an element of that set. No basic open set contains x and does not contain y. Therefore, any set which contains x will also contain y and the union of that set with some set containing y will not be the null set, as it contains at least the element y.
III. This is second countable.
For the basis, Θ, defined by the basic open sets where $a, b \in \Re$, there is an equivalent and countable basis, Φ, with values which are elements of the rational numbers. We must consider all three forms of that the basic open sets take. 1. $\forall [-1, b_1) \subset \Theta$, $\exists$ some $b_2 < b_1$ where b_2 is rational, by the density of rational numbers. $\exists b_3 < b_2$, $b_3 \in \Re$, by the density of reals, so that $[-1, b_3) \subset [-1, b_2) \subset [-1, b_1)$ with

$[-1, b_3), [-1, b_1) \subset \Theta$, $[-1, b_2) \subset \Phi$. 2. $\forall (a_1, 1] \subset \Theta$, $\exists$ some $a_2 > a_1$ where a_2 is rational, by the density of rational numbers. $\exists a_3 > a_2$, $a_3 \in \Re$, by the density of reals, so that $(a_3, 1] \subset (a_2, 1] \subset (a_1, 1]$ with $(a_3, 1], (a_1, 1] \subset \Theta$, $(a_2, 1] \subset \Phi$. 3. $\forall (b_1, a_1) \subset \Theta$, $\exists$ some a_2, b_2, which are rational and $b_1 > a_2 > a_1$, $a_2 < b_2 < b_1$, by the density of rational numbers. $\exists a_3, b_3 \in \Re$ where $b_2 > a_3 > a_2$ and $a_3 < b_3 < b_2$, by the density of reals, so that $(b_3, a_3) \subset (b_2, a_2) \subset (b_1, a_1)$ with $(b_3, a_3), (b_1, a_1) \subset \Theta$, $(b_2, a_2) \subset \Phi$. Therefore, the two bases are equivalent and, since Φ is countable as all of the sets are defined with rational numbers, the topology is second countable.
IV. This is first countable.
For any $x \in [-1, 1]$, let $x \in U$, an open set in the topological space. Let $B_x = \{\theta \in \Theta : x \in \theta\}$ be the neighborhood basis for the point x. Θ is the countable basis for the topological space. $\exists \theta \in \Theta$ with $x \in \theta \subset U$ and $\theta \in B_x$. Since the intersection of all $\theta \in B_x$ contains x, Θ is a neighborhood basis at x. There is a countable neighborhood basis for any point $x \in X$, therefore the topological space is first countable.
■

Problem 1.B.7 *Let $X = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and define an order on X by declaring $(a, b) \leq (u, v)$ if and only if $a < u$, or $a = u$ and $b \leq v$. Show that this is a linear ordering and determine of X (with the order topology) is T_2, first countable, or second countable.*

Proof
In order for $\leq$ to be a linear ordering, for $x, y \in X$, either $x \leq y$ or $y \leq x$. Take (a, b) and $(u, v) \in X$. Assume without loss of generality that $(a, b) \not\leq (u, v)$. Then 1) $a \geq u$ and 2) either $a \neq u$ or $b > v$.

Case 1. If $a \geq u$ and $a \neq u$ then $u < a$ which implies by definition that $(u, v) \leq (a, b)$
Case 2. If $a > u$ and $b \geq v$ then $u < a$ which implies by definition that $(u, v) \leq (a, b)$
Case 3. If $a = u$ and $b \geq v$ then $v \leq b$ which implies by definition that $(u, v) \leq (a, b)$.
Since $(a, b) \not\leq (u, v)$ implies $(u, v) \leq (a, b)$, $\leq$ is a linear order on X.

X is T_2. To see this take $(x_1, x_2), (y_1, y_2) \in X$ with $(x_1, x_2) \neq (y_1, y_2)$. We must consider two cases.
Case 1. If $x_1 = y_1$, then define $d = |x_2 - y_2|$. Then consider the open intervals (as defined in the definition of the order topology)

$$I_1 = ((x_1, \max\{x_2 - d/3, 0\}), (x_1, \min\{x_2 + d/3, 1\}))$$

and

$$I_2 = ((y_1, \max\{y_2 - d/3, 0\}), (y_1, \min\{y_2 + d/3, 1\}))$$

Then $I_1 \bigcap I_2 = \emptyset$, $(x_1, x_2) \in I_1$, and $(y_1, y_2) \in I_2$.
Case 2. If $x_1 \neq y_1$, then define $\hat{d} = |x_1 - y_1|$, Then consider the open intervals

$$I_3 = ((\max\{x_1 - \hat{d}/3, 0\}, x_2), (\min\{x_1 + \hat{d}/3, 1\}, x_2))$$

and

$$I_4 = ((\max\{y_1 - \hat{d}/3, 0\}, y_2), (\min\{y_1 + \hat{d}/3, 1\}, y_2))$$

Then $I_3 \bigcap I_4 = \emptyset$, $(x_1, x_2) \in I_3$, and $(y_1, y_2) \in I_4$.
Since distinct points in X can be separated by two distinct open sets, X is T_2.

X is second countable. Let $((x_1, x_2), (y_1, y_2)) = \{c \in X | (x_1, x_2) < c < (y_1, y_2); x_1, x_2, y_1, y_2 \in \mathbb{Q}\}$. Sets of the form $((x_1, x_2), (y_1, y_2))$ together with sets of the form $\{c \in X | c < (z_1, z_2); z_1, z_2 \in \mathbb{Q}\}$ and sets of the form $\{c \in X | (z_1, z_2) < c; z_1, z_2 \in \mathbb{Q}\}$ constitute a countable basis for a topology on X which is equivalent to the order topology on X. Therefore X is second countable.

X is first countable since by Problem $1.B.1$, every second countable set is first countable.

■

Problem 1.B.8

Problem 1.B.9 *Suppose X is first countable (respectively, second countable), Y is a topological space and $f : X \to Y$ is continuous, open (i.e., f takes open sets to open sets), and onto. Show Y is first countable (respectively, second countable).*

Proof X is both first countable and second countable, while Y is a topological space. $f : X \to Y$ is continuous, open and onto. There are countably many open sets $\hat{U} \in U, \hat{U} \subset X$. Since f is continuous and open, it maps every open set $\hat{U} \to \hat{V}$ where $\hat{V} \in V, \hat{V} \subset Y$. Since f is onto, it maps to every $\hat{V} \subset Y$ and every $\hat{U} \subset X$ maps onto no more than one $\hat{V} \subset Y$. Therefore, the number of open sets in Y is at most equal to the number of open sets in X, and since the later is countable, so is the former. $\forall \hat{B}_Y \in B_Y$, the basis of Y, $\exists \hat{B}_X \in B_X : f(\hat{B}_X) = \hat{B}_Y$. Since the basis of X is a countable set and f is onto, the basis of Y must be a countable set. Therefore, Y is second countable.
$\forall y \in Y$, $\exists V \subset Y : y \in V$. $\exists x \in X$, $U \subset X : x \in U$. Since X is first countable, x has a countable neighborhood basis, so $\exists \theta_x \in \Theta_x$ where $x \in \theta_x \subset U$. So there are $f(\theta_x) \to \phi_y$, which forms the neighborhood basis for $y \in Y$. So, $\Phi_y = \{f(\theta_x) : x \in X, \theta_x \in \Theta_x\}$. Therefore, Y is first countable.

■

Problem 1.B.10

- *Suppose $(X, \mathcal{U})$ is first countable, $(Y, \mathcal{V})$ is a topological space, and $f : X \to Y$ is continuous, open and onto. Show $(Y, \mathcal{V})$ is first countable.*
- *Suppose $(X, \mathcal{U})$ is second countable, $(Y, \mathcal{V})$ is a topological space, and $f : X \to Y$ is continuous, open and onto. Show $(Y, \mathcal{V})$ is second countable.*

Proof

- Since f is onto, given $y \in Y$, $\exists x \in X$ such that $f(x) = y$. Since $(X, \mathcal{U})$ is first countable, $\exists$ a countable neighborhood basis for $\mathcal{U}$ at x, $\mathcal{B}_x$. Define $\hat{\mathcal{B}}_y = \{f(B) | B \in \mathcal{B}_x\}$. We will show $\hat{\mathcal{B}}_y$ is a countable neighborhood basis for $\mathcal{V}$ at y. Clearly $\hat{\mathcal{B}}_y$ is countable since $\mathcal{B}_x$ is countable. The elements of $\hat{\mathcal{B}}_y$ are open in Y since f is open. Also $y = f(x) \in \bigcap_{B \in \hat{\mathcal{B}}_y} B$. Now given $V \in \mathcal{V}$ with $y \in V$, by continuity $\exists\, U \in \mathcal{U}$ such that $x \in U$ and $f(U) \subset V$. Since $\mathcal{B}_x$ is a neighborhood basis of x, $\exists\, B \in \mathcal{B}_x$ such that $x \in B \subset U$. Therefore $y = f(x) \in f(B) \subset f(U) \subset V$. Since $f(B) \in \hat{\mathcal{B}}_y$, $\hat{\mathcal{B}}_y$ is by definition a neighborhood basis for $\mathcal{V}$ at y. Since $\hat{\mathcal{B}}_y$ is countable, and y is arbitrary in $(Y, \mathcal{V})$, $(Y, \mathcal{V})$ is first countable.
- Since $(X, \mathcal{U})$ is second countable, $\exists$ basis $\mathcal{B}$ of $(X, \mathcal{U})$ which is countable. Define $\hat{\mathcal{B}} = \{f(B) | B \in \mathcal{B}\}$. Clearly this is countable since $\mathcal{B}$ is countable. This is also a set of open sets since f is open. Take $y \in Y$ and take $V \in \mathcal{V}$ such that $y \in V$. Since f is onto, $\exists\, x \in X$ such that $f(x) = y$. By continuity, $\exists\, U \in \mathcal{U}$ with $x \in U$ such that $f(U) \subset V$. Since $\mathcal{B}$ is a basis of $(X, \mathcal{U})$, $\exists\, B \in \mathcal{B}$ such that $x \in B \subset U$. This implies $y = f(x) \in f(B) \subset f(U) \subset V$. Since $f(B) \in \hat{\mathcal{B}}$, $\hat{\mathcal{B}}$ is a basis for $(Y, \mathcal{V})$.

■

1.C Metric Spaces: A Preview

Problem 1.C.1 *Supose (X,d) is a metric space and p is a point of X. Show that for each $r > 0$,$\{x \in X : d(x,p) > r\}$ is an open set.*

Proof We want to show for $r > 0$, $\{x \in X | d(x,p) > r\}$ is the union of basic open sets. Take $y \in X$ such that $d(y,p) > r$. Then define $\mu = d(y,p) - r$. Examine $S_\mu(y)$. This is a basic open set and $\forall\, z \in S_\mu(y)$, $d(z,p) > r$ by the triangle inequality. The union of all sets of this form is open and is equal to the set $\{x \in X | d(x,p) > r\}$. Thus for each $r > 0$, $\{x \in X | d(x,p) > r\}$ is open, and the proof is complete.

Alternate Proof
(X,d) is a metric space and $p \in X$. For any $r > 0$, let $U = \{x \in X : d(x,p) > r\}$. Let $V = X \setminus U$, so that $V = \{x \in X : d(x,p) \leq r\}$. The set of points $R = \{x \in X : d(x,p) = r\}$ makes up the frontier of V. For any open set or basic open set containing an element $a \in R$, the set must contain at some element $b \in \{x \in X : d(x,p) > r\}$ with $b \in \{x \in X : d(x,a) < \epsilon\}, \epsilon > 0$. The set will also contain some element $c \in \{x \in X : d(x,p) < r\}$ with $c \in \{x \in X : d(x,a) < \epsilon\}, \epsilon > 0$. Therefore, R is the frontier of V and by the definition of V, $R \subset V$. Therefore, V is a closed set and $U = X \setminus V$ is an open set in the topology.
■

Problem 1.C.2

Problem 1.C.3

Problem 1.C.4

Problem 1.C.5 *Let $X = \Re^2$ and d be the usual metric. Denote $(0,0)$ by O. Define $\hat{d} : X \times X \to [0,\infty)$ by $\hat{d}(p,q) = d(O,p) + d(O,q)$ for $p,q \in X$ and $p \neq q$, and $\hat{d}(p,p) = 0$ for all $p \in X$.*

Proof Part A. $\hat{d}$ is a metric. I. $\hat{d}(p,p) = 0$ for all $p \in X$ and for any two distinct points $x, y \in X$, at least one of them is not equal to O. Using the property of the usual metric where $d(x,O) = 0$ only when $x = O$ tells us that at least one of the two elements in the sum of $\hat{d}$ will be non-zero and, since all $d(x,y) \geq 0$, $\hat{d} > 0$. Therefore, $\hat{d}(x,y) = 0$ only when $x = y$.
II.

$$\begin{aligned}\hat{d}(x,y) &= d(O,x) + d(O,y) \\ &= d(O,y) + d(O,x) \\ &= \hat{d}(y,x)\end{aligned}$$

III.

$$\begin{aligned}\hat{d}(x,y) + \hat{d}(y,z) &= d(O,x) + d(O,y) + d(O,y) + d(O,z) \\ &\geq d(O,x) + d(O,z) \forall x, y \in \Re, d(x,y) \geq 0 \\ &= \hat{d}(x,z)\end{aligned}$$

Part B. Show that all points other than O are open. For any point $x \in \Re^2$, we can create a open set out of basic open sets in the basis $B = \left\{S_\epsilon^{\hat{d}} : x \in X, \epsilon > 0\right\}$, where $S_\epsilon^{\hat{d}}(x) = \{y \in X : d(O,x) + d(O,y) < \epsilon\}$ and thus every point is open.
Part C. Describe the basis neighborhoods whose center is at O. For any open set $S_\epsilon^{\hat{d}} = \{x \in X : d(O,O) + d(O,x)$ which is a basic open set centered at O, the neighborhood of these sets is the set which, for some open set U, has the property that $\forall S_\epsilon^{\hat{d}}$ centered at O, $S_\epsilon^{\hat{d}} \subset U$. ■

Problem 1.C.6 *If X is a set, a function $d : X \times X \to [0, \infty)$ is called a pseudo-metric if and only if*
(i) $d(x, x) = 0$ for all $x \in X$.
(ii) $d(x, y) = d(y, x)$ for all $(x, y) \in X \times X$.
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.
Suppose $X = \{f | f : \mathcal{E}^1 \to \mathcal{E}^1$ and f is integrable on $[0, 1]\}$. Define $d : X \times X \to [0, \infty)$ by $d(f, g) = \int_0^1 |f(x) - g(x)|dx$. Show that d is a pseudo-metric but not a metric. Is there a theorem analogous to (1.C.3) for pseudo-metric spaces?

Proof First we show d is a pseudo-metric by verifying the three properties. (i) holds since for $f \in X$,

$$\begin{aligned} d(f, f) &= \int_0^1 |f(x) - f(x)|dx \\ &= \int_0^1 0dx \\ &= 0 \end{aligned}$$

(ii) holds since for $f, g \in X$,

$$\begin{aligned} d(f, g) &= \int_0^1 |f(x) - g(x)|dx \\ &= \int_0^1 |-1(f(x) - g(x))|dx \\ &= \int_0^1 |g(x) - f(x)|dx \\ &= d(g, f). \end{aligned}$$

(iii) holds since, by the triangle inequality for real numbers, for $f, g, h \in X$,

$$\begin{aligned} d(f, h) &= \int_0^1 |f(x) - h(x)|dx \\ &= \int_0^1 |f(x) - g(x) + g(x) - h(x)|dx \\ &\leq \int_0^1 |f(x) - g(x)| + |g(x) - h(x)|dx \\ &= \int_0^1 |f(x) - g(x)|dx + \int_0^1 |g(x) - h(x)|dx \\ &= d(f, g) + d(g, h) \end{aligned}$$

Therefore X is a pseudo-metric space. X is not, however, a metric space. To see this, define

$$f(x) = \begin{cases} 0 & 0 \leq x \leq 1, x \neq 1/2 \\ 2 & x = 1/2 \end{cases},$$

and define

$$g(x) = 0 \qquad 0 \leq x \leq 1.$$

Note that

$$d(f, g) = \int_0^1 |f(x) - g(x)|dx = 0$$

but $f \neq g$. But the definition of a pseudo-metric space states that if $d(x,y) = 0$, then $x = y$. This implication is not true in this case, so X is not a metric space.

There is a theorem analogous to theorem (1.C.3) for pseudo-metric spaces. That is, if (X,d) is a pseudo-metric space, then $\mathcal{B} = \{S_\epsilon(x) | x \in X, \epsilon > 0\}$ is a basis for a topology on X.
We will use theorem 1.A.6. First we note that clearly for $\epsilon > 0$, $\bigcup_{x \in X} S_\epsilon(x) = X$ since $x \in S_\epsilon(x)$. Now we will show, given $\epsilon > 0, \mu > 0$, for $p \in S_\epsilon(x) \bigcap S_\mu(y)$ then $\exists\, \nu > 0$ such that $S_\nu(p) \subset S_\epsilon(x) \bigcap S_\mu(y)$. Since $d(x,y) = 0$ does not imply $x = y$, we have more cases to consider.

Case 1. $d(x,y) \neq 0$, $d(x,p) \neq 0$, $d(y,p) \neq 0$. The proof proceeds as in 1.C.3.

Case 2. $d(x,y) \neq 0$, but either $d(x,p) = 0$ or $d(y,p) = 0$. Note that $d(x,p) = d(y,p) = 0$ is not possible since the triangle inequality states $d(x,y) \leq d(x,p) + d(p,y) = 0$, but $d(x,y) > 0$, so we have a contradiction. Without loss of generality let $d(y,p) = 0$. Then $d(x,p) = \lambda$. Let $\nu < \min\{\mu - \lambda, \epsilon\}$. Then $p \in S_\nu(p) \subset S_\epsilon(x) \bigcap S_\mu(y)$.

Case 3. Take $d(x,y) = d(y,x) = 0$. We note that for $c \in X$, by the triangle inequality $d(x,c) \leq d(x,y) + d(y,c) = d(y,c)$. Similarly $d(y,c) \leq d(x,c)$, so we have $d(x,c) = d(y,c)$. Since $d(x,y) = d(y,x) = 0$, for $\epsilon > 0$ and $\mu > 0$, then $x \in S_\epsilon(x)$ and $y \in S_\epsilon(y) = S_\epsilon(x)$. Similarly $x \in S_\mu(x) = S_\mu(y)$ and $y \in S_\mu(y)$. Then $S_\epsilon(x) \bigcap S_\mu(y) = S_{\min\{\mu,\epsilon\}}(x) = S_{\min\{\mu,\epsilon\}}(y)$. Now take $p \in S_{\min\{\mu,\epsilon\}}(x)$. If $d(x,p) = 0$ we are done since, if we take $\eta = \min\{\mu, \epsilon\}$, $p \in S_\eta(x) = S_\eta(p) \subset S_\epsilon(x) \bigcap S_\mu(y)$. Therefore assume $d(x,p) = d(y,p) = \lambda > 0$. Let $\nu < \min\{\mu - \lambda, \epsilon - \lambda\}$, then $p \in S_\nu(p) \subset S_{\min\{\mu,\epsilon\}}(x)$.

In all cases we showed for $p \in S_\epsilon(x) \bigcap S_\mu(y)$ then $\exists\, \nu > 0$ such that $S_\nu(p) \subset S_\epsilon(x) \bigcap S_\mu(y)$, so the proof is complete. ■

Problem 1.C.7 *Let* $X = \Re^2$*, and for points* $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$ *define* $d(x,y) = \{1, x_1 \neq x_2, y_1 \neq y_2 0, x_1$
Show that d is a metric and that the sets shown in the textbook figure possess different area if d is used to measure the length of sides. Describe the topology induced by this metric.

Proof I. $d(x,x) = 0$ for all $x \in X$, with $x = (x_1, x_2)$, $x_1, x_2 \in \Re$. This meets the requirements set out in the hypothesis. If either the first or second element is different between the two points measured by the metric, then the metric will return a non-zero value. Therefore, $d(x,y) = 0$ only when $x = y$.

II. Switching the order or the points will only switch the order in which they appear in the definition of d, therefore $d(x,y) = d(y,x)$.

III. $d(x,y) + d(y,z)$ can have nine possible cases, where each can be either 1, 1/2 or 0. Where the sum equals 1 or greater, we can exploit the fact that $d(x,z) \leq 1$. Of the three remaining cases, where both equal 0, property I shows that $x = y$ and $y = z$. Therefore, $x = z$ and $d(x,z) = 0$. If $d(x,y) = 0, d(y,z) = 1/2$, then $x = y$ and therefore $d(x,z) = d(y,z) = 1/2$. If $d(x,y) = 1/2, d(y,z) = 0$, then $y = z$ and therefore $d(x,z) = d(x,y) = 1/2$. Therefore, $d(x,z) \leq d(x,y) + d(y,z)$.

The figure depicted on the left shows four segments where either $x_1 = x_2$ or $y_1 = y_2$. $d(x,y) = 1/2$ for all of these segments. On the right figure, for each segment, $x_1 \neq x_2$ and $y_1 \neq y_2$. $d(x,y) = 1$ for all of these segments. While computing area has not been rigorously defined, we could reason that the quadrilateral with all sides greater in a metric space must have greater area.
$B = \left\{S_\epsilon^d : x \in X, \epsilon > 0\right\}$, with basic open sets $S_\epsilon^d(x) = \{y \in X : d(x,y) < \epsilon\}$ forms a basis for this topology. If we choose some arbitrarily small ϵ, which can be less than 1/2, basic open sets are the singletons

which contain only their center. If ϵ is allowed to be greater than 1/2, we have basic open sets which are two lines intersecting at the center of the set. If $\epsilon > 1$, then a basic open set is the whole space. However, if we consider the basic open sets with arbitrarily small ϵ, basic open sets of singletons yields a topology where open sets are collections of discrete points, which can be infinitely unioned to give open sets with any collection of points. The topology is therefore equal to $P(x)$ unioned with the whole set and null set, which is the discrete topology.
■

Problem 1.C.8

Problem 1.C.9

1.D Building New Spaces from Old Ones

Problem 1.D.1 *Determine the relative topology that is induced by $\mathcal{E}^1$ on the integers.*

Proof The relative topology that is induced by $\mathcal{E}^1$ on the integers is the discrete topology. To show this, we show every singleton is an open set in the relative topology. For $z \in \mathbb{Z}$, take the open set $(z - 1/2, z + 1/2) \in \mathcal{E}^1$. Then $\{z\} = \mathbb{Z} \bigcap (z - 1/2, z + 1/2)$. By the definition of an open set in the relative topology, the singleton $\{z\}$ is an open set in the relative topology that is induced by $\mathcal{E}^1$ on the integers. Since z is arbitrary, every singleton is open in this topology. Therefore the relative topology that is induced by $\mathcal{E}^1$ on the integers is the discrete topology.
■

Problem 1.D.2

Problem 1.D.3

Problem 1.D.4 *Show that if $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are T_2 spaces, then so is $X \times Y$*

Proof Take $(x_1, y_1), (x_2, y_2) \in X \times Y$ with $(x_1, x_2) \neq (y_1, y_2)$. We want to show that $(x_1, y_1), (x_2, y_2)$ can be seperated into distinct open sets in the product topology. We must consider two cases.

Case 1. Suppose without loss of generality that $x_1 = x_2$ but $y_1 \neq y_2$. Since Y is T_2, $\exists\ V_1, V_2 \in \mathcal{V}$ such that $y_1 \in V_1$, $y_2 \in V_2$ and $V_1 \bigcap V_2 = \emptyset$. Now let $U \in \mathcal{U}$ such that $x \in U$ and consider the (basic) open sets $U \times V_1$ and $U \times V_2$ in the product topology. We have $(x_1, y_1) \in U \times V_1$ and $(x_2, y_2) \in U \times V_2$ with $(U \times V_1) \bigcap (U \times V_2) = \emptyset$ since $V_1 \bigcap V_2 = \emptyset$. A similar proof holds if $x_1 \neq x_2$ but $y_1 = y_2$.

Case 2. Suppose that $x_1 \neq x_2$ and $y_1 \neq y_2$. Since X is T_2, $\exists\ U_1, U_2 \in \mathcal{U}$ such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \bigcap U_2 = \emptyset$. Similarly, since Y is T_2, $\exists\ V_1, V_2 \in \mathcal{V}$ such that $y_1 \in V_1$, $y_2 \in V_2$ and $V_1 \bigcap V_2 = \emptyset$. Now consider the (basic) open sets $U_1 \times V_1$ and $U_2 \times V_2$ in the product topology. We have $(x_1, y_1) \in U_1 \times V_1$ and $(x_2, y_2) \in U_2 \times V_2$ with $U_1 \times V_1 \bigcap U_2 \times V_2 = \emptyset$ since $U_1 \bigcap U_2 = \emptyset$ and $V_1 \bigcap V_2 = \emptyset$.

These two cases show $X \times Y$ is T_2. ■

Problem 1.D.5

Problem 1.D.6

Problem 1.D.7

Problem 1.D.8

Problem 1.D.9

Problem 1.D.10

Problem 1.D.11

Problem 1.D.12

1.E A Potporri of Fundamental Concepts

Problem 1.E.1

Problem 1.E.2

Problem 1.E.3

Problem 1.E.4

Problem 1.E.5

Problem 1.E.6

Problem 1.E.7

Problem 1.E.8

Problem 1.E.9 *Suppose X is a topological space and $A \subset B \subset X$. Find an example where $intA$ in B is not the same as $intA$ in X.*

Proof Let $X = \mathcal{E}^1$ and $B = [0, 1)$ with the relative topology and let $A = [0, 1/2)$. A is open in the relative topology since $(-1, 1/2)$ is open in $\mathcal{E}^1$ and $(-1, 1/2) \bigcap [0, 1) = [0, 1/2) = A$. We claim $[0, 1/2) = intA$ in B. Since $intA \subset A$ by the definition of interior, we are left to show that $A \subset intA$. Since $intA$ is the union of all open sets containing A, and A is an open set in the relative topology containing A, $A \subset intA$, and thus $A = intA$ in B. But $(0, 1/2) = intA$ in X. Since $[0, 1/2) \neq (0, 1/2)$, we have found an example where $intA$ in B is not the same as $intA$ in X.

■

Problem 1.E.10

Problem 1.E.11

Problem 1.E.12 *Let $X = \{a, b, c\}$ and let $\mathcal{U} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ be a topology for X. Find the derived set of each subset of X.*

Proof We claim every subset of X except for $\{c\}$ has a derived set equal to $\{c\}$. Take any subset A of X which is not $\{c\}$. Then $A\backslash\{a\}\bigcap\{a\} = \emptyset$ since any set without a has an empty intersection with a. Since $\{a\}$ is an open set, then a is not an accumulation point of A. Similarly $A\backslash\{b\}\bigcap\{b\} = \emptyset$ shows that b is not an accumulation point of A since $\{b\}$ is an open set. Now since X is the only open set containing c, we examine $A\backslash\{c\}\bigcap X$ to determine if c is an accumulation point of A. Since $A \neq \{c\}$, $A\backslash\{c\} \neq \emptyset$. This implies $A\backslash\{c\}\bigcap X \neq \emptyset$. Thus c is an accumulation point of A.

Now we claim $\{c\}' = \emptyset$. By the above arguments, we know a and b are not accumulation points of $\{c\}$. Now we show c is not an accumulation point of $\{c\}$. We note $\{c\}\backslash\{c\}\bigcup X = \emptyset$ since $\{c\}\backslash\{c\} = \emptyset$. Thus c is not an accumulation point of $\{c\}$ by the definition of accumulation point, and thus $\{c\}' = \emptyset$.
■

Problem 1.E.13

Problem 1.E.14

Problem 1.E.15

Problem 1.E.16

Problem 1.E.17

Problem 1.E.18

Problem 1.E.19

Problem 1.E.20

Problem 1.E.21

Problem 1.E.22

Problem 1.E.23

Problem 1.E.24

Problem 1.E.25 *Prove that in a T_1 space, a point x is an accumulation point of a set A if and only if every neighborhood of x contains infinitely many points of A.*

Proof Suppose every neighborhood of x contains infinitely many points of A. Then since every open set is a neighborhood of x, then every open set containing x contains infinitely many points of A. Then for every open set U, $U\bigcap(A\backslash\{x\}) \neq \emptyset$, since U has infinitely many points of $A\backslash\{x\}$.

Suppose x is an accumulation point of a T_1 space $(X,\mathcal{U})$. Then $\forall\ U \in \mathcal{U}$, $U\bigcap(A\backslash\{x\}) \neq \emptyset$. Assume on the contrary that $\exists$ some neighborhood N of x which contains finitely many points of A. Since N is a neighborhood of x, $\exists\ U \in \mathcal{U}$ such that $x \in U \subset N$. This set U must also contain finitely many points, say n, of A. Since X is T_1, then for each $y_i \in A$, $1 \leq i \leq n$, $\exists\ U_i \in \mathcal{U}$ such that $x \in U_i$, but $y_i \notin U_i$. Then $V = (\bigcap_{i=1}^{n} U_i)\bigcap U$ is an open set, since the finite intersection of open sets is open. We note $V \subset U$ and $y_i \notin V$ for $1 \leq i \leq n$. Then $V\bigcap(A\backslash\{x\}) = \emptyset$, which is a contradiction to x being an accumulation point of A. Therefore every neighborhood of x must contain infinitely many points of A.
■