Homework Key $3.5, 3.6$

Section 3.5

#37:

$f(x) = \left[x + (2x)^{1/2}\right]^{1/3}$

$$\begin{aligned} f'(x) &= \frac{1}{3}\left[x + (2x)^{1/2}\right]^{-2/3} \frac{d}{dx}\left[x + (2x)^{1/2}\right] \\ &= \frac{1}{3}\left[x + (2x)^{1/2}\right]^{-2/3} \left(1 + \frac{d}{dx}\left[(2x)^{1/2}\right]\right) \\ &= \frac{1}{3}\left[x + (2x)^{1/2}\right]^{-2/3} \left(1 + 1/2\,(2x)^{-1/2}\left(\frac{d}{dx}[2x]\right)\right) \\ &= \frac{1}{3}\left[x + (2x)^{1/2}\right]^{-2/3} \left(1 + 1/2\,(2x)^{-1/2}(2)\right) \\ &= \frac{1}{3}\left[x + (2x)^{1/2}\right]^{-2/3} \left(1 + (2x)^{-1/2}\right) \end{aligned}$$

#48:

$h'(x) = f'(g(x))\,g'(x)$

All three parts require we estimate the value of $h'(x)$ for given x. The estimation technique can be done in many ways but always boils down to estimating the slope of a tangent line to the graph of f and the slope of a tangent line to the graph of g. For example in part (a) we need to know the slope of the tangent line to f when $x = 0$ and the slope of the tangent to g when $x = -1$:

$$\begin{aligned} h'(-1) &= f'(g(-1))\,g'(-1) \\ &= f'(0)\,g'(-1) \end{aligned}$$

#69:

This problem is setting us up to know what the "backwards" derivatives of $\tan(x)$ and $\sec(x)$ are. That is, if you want to know what functions have the property that their derivatives equal $\sec(x)$ then you now know that $f(x) = \ln|\sec(x) + \tan(x)|$ is one such function.

$$\begin{aligned} f(x) &= \ln|\cos(x)| \\ f'(x) &= \frac{1}{\cos(x)}\frac{d}{dx}[\cos(x)] \\ &= \frac{1}{\cos(x)}(-\sin(x)) \\ &= -\tan(x) \end{aligned}$$

and

$$\begin{aligned}
f(x) &= \ln|\sec(x)+\tan(x)| \\
f'(x) &= \frac{1}{\sec(x)+\tan(x)}\frac{d}{dx}[\sec(x)+\tan(x)] \\
&= \frac{1}{\sec(x)+\tan(x)}\left(\sec(x)\tan(x)+\sec^2(x)\right) \\
&= \frac{1}{\sec(x)+\tan(x)}(\tan(x)+\sec(x))\sec(x) \\
&= \sec(x)
\end{aligned}$$

Section 3.6

#12:

$$\begin{aligned}
e^{xy}+1 &= x^2 \\
\frac{d}{dx}[e^{xy}+1] &= 2x \\
e^{xy}\left[(1)y+x\frac{dy}{dx}\right]+0 &= 2x \\
y+x\frac{dy}{dx} &= \frac{2x}{e^{xy}} \\
x\frac{dy}{dx} &= \frac{2x}{e^{xy}}-y \\
\frac{dy}{dx} &= \frac{2}{e^{xy}}-\frac{y}{x}
\end{aligned}$$

#36

Find the equation of the tangent line to the graph of $3^x+\log_2(xy)=10$ at the point $(2,1)$. First note that $(2,1)$ is on the graph since $3^2+\log_2(2)=9+1=10$

$$\begin{aligned}
3^x+\log_2(xy) &= 10 \\
\frac{d}{dx}[3^x+\log_2(xy)] &= \frac{d}{dx}[10] \\
3^x\ln(3)+\frac{1}{xy\ln(2)}\frac{d}{dx}[xy] &= 0 \\
\frac{1}{xy\ln(2)}\frac{d}{dx}[xy] &= -3^x\ln(3) \\
\frac{1}{xy\ln(2)}\left[y+x\frac{dy}{dx}\right] &= -3^x\ln(3) \\
y+x\frac{dy}{dx} &= [-3^x\ln(3)][xy\ln(2)] \\
x\frac{dy}{dx} &= [-3^x\ln(3)][xy\ln(2)]-y
\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{[-3^x \ln(3)][xy\ln(2)] - y}{x} \\ \left.\frac{dy}{dx}\right|_{(2,1)} &= \frac{[-3^2 \ln(3)][2\ln(2)] - 1}{2} \\ &\approx -7.35\end{aligned}$$

So the equation of the tangent line at $(2,1)$ is approximately

$$y - 1 = -7.35\,(x-2)$$

#68

Find the equation of the tangent line to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0). First note that since (x_0, y_0) is on the graph then $\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1$. We'll use this later.

$$\begin{aligned}\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 \\ \frac{d}{dx}\left[\frac{x^2}{a^2} - \frac{y^2}{b^2}\right] &= 0 \\ \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} &= 0 \\ \frac{2x}{a^2} &= \frac{2y}{b^2}\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{2x/a^2}{2y/b^2} \\ &= \frac{x/a^2}{y/b^2} \\ \left.\frac{dy}{dx}\right|_{(x_0,y_0)} &= \frac{x_0/a^2}{y_0/b^2}\end{aligned}$$

So the tangent line at (x_0, y_0) can be written as

$$\begin{aligned}y - y_0 &= \frac{x_0/a^2}{y_0/b^2}(x - x_0) \\ \frac{y-y_0}{x-x_0} &= \frac{x_0/a^2}{y_0/b^2} \\ (y-y_0)\frac{y_0}{b^2} &= (x-x_0)\frac{x_0}{a^2} \\ \frac{yy_0}{b^2} - \frac{y_0^2}{b^2} &= \frac{xx_0}{a^2} - \frac{x_0^2}{a^2} \\ \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} &= \frac{xx_0}{a^2} - \frac{yy_0}{b^2} \\ 1 &= \frac{xx_0}{a^2} - \frac{yy_0}{b^2} \text{ because the point } (x_0, y_0) \text{ is on the graph. (See above.)}\end{aligned}$$