Mathematics 258-A — Spring 2006

Exam 2

February 16, 2006

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Name

Directions:

- **Only write on one side of each page.**
- **Use terminology correctly.**
- **Show your work: answers that can be obtained from a calculator will not receive credit.**
- **Partial credit is awarded for correct approaches so justify your steps.**

Do any seven (7) of the following.

1. [14 points] If $\lim_{x\to a} f(x) = -4$ and $\lim_{x\to a} g(x) = 4$, compute the limits that exist and for any that don't exist, explain why.

   (a) $\lim_{x\to a} \frac{2f(x)-3g(x)}{[f(x)]^2}$

      i. $= \frac{2(-4)-3(4)}{(-4)^2} = -\frac{5}{4}$

   (b) $\lim_{x\to a} \frac{1}{f(x)+g(x)}$.

      i. This limit does not exist since dividing the number 1 by numbers that are closer and closer to zero results in unbounded behavior. More specifically, $\frac{1}{f(x)+g(x)}$ is unbounded as x gets closer and closer to a.

2. [7, 7 points] If $g(x) = x^2 + 3x + 5$

   (a) Find $\frac{g(x+h)-g(x)}{h}$

      i. $= \frac{\left((x+h)^2+3(x+h)+5\right)-\left(x^2+3x+5\right)}{h} = \frac{(x+h)^2+3h-x^2}{h} = \frac{h(2x+h+3)}{h}$

   (b) Find $g'(x)$ by carefully evaluating $\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$

      i. $= \lim_{h\to 0} \frac{h}{h}(2x+h+3) = 2x+3$.

3. [14 points] Differentiate the following.

   (a) $f(x) = 2x^5 - 7x^3 + x + 5 + x^{-2}$

      i. $f'(x) = 10x^4 - 21x^3 + 1 + 0 - 2x^{-3}$

   (b) $f(x) = \frac{1}{\sqrt{x^4+9}} = \left(x^4+9\right)^{-1/2}$

      i. $f'(x) = (-1/2)\left(x^4+9\right)^{-3/2}\left(4x^3+0\right)$

   (c) $f(x) = (\sqrt{x}+1)^{3/2} = \left(x^{1/2}+1\right)^{3/2}$

      i. $f'(x) = \frac{3}{2}\left(x^{1/2}+1\right)^{1/2}\left(\frac{1}{2}x^{-1/2}+0\right)$

4. [14 points] Evaluate the following:

(a) $\frac{dT}{dt}$ where $T = (2t+3)^5 + t^3$

   i. $\frac{dT}{dt} = 5\,(2t+3)^4\,(2) + 3t^2$

(b) $\frac{d}{dv}\left(\frac{1}{3\sqrt{v}}\right)$

   i. $= \frac{d}{dv}\left(\frac{1}{3}v^{-1/2}\right) = \frac{1}{3}\left(-\frac{1}{2}\right)v^{-3/2}.$

(c) $\frac{d}{dy}\left(y^5 - y^2\right)\Big|_{y=2}$

   i. $\frac{d}{dy}\left(y^5 - y^2\right) = 5y^4 - 2y$ so we have $\frac{d}{dy}\left(y^5 - y^2\right)\Big|_{y=2} = 5\left(2^4\right) - 2\,(2) = 76$

(d) $\frac{d^2}{dx^2}\left(2x^3+3\right)^2\Big|_{x=-1}$

   i. $\frac{d}{dx}\left(2x^3+3\right)^2 = 2\left(2x^3+3\right)^1\left(6x^2\right) = 24x^5 + 36x^2$ so $\frac{d^2}{dx^2}\left(2x^3+3\right)^4 = \frac{d}{dx}\left(24x^5+36x^2\right) = 120x^4 + 72x$ and finally $\frac{d^2}{dx^2}\left(2x^3+3\right)^2\Big|_{x=-1} = 120\,(-1)^4 + 72\,(-1) = 48$

5. [14 points] A helicopter is rising at the rate of 32 feet per second. At a height of 128 feet the pilot accidentally drops a pair of binoculars. After t seconds, the binoculars have height $s(t) = -16t^2 + 32t + 128$ feet from the ground. How fast will they be falling when they hit the ground?

   (a) The binoculars hit the ground at the time t when $s(t) = 0$. So we solve
   $-16t^2 + 32t + 128 = 0$
   $(-16)\left(t^2 - 2t - 8\right) = 0$
   $(x-4)(x+2) = 0.$
   So the values of t for which $s(t) = 0$ are $t = 4$ and $t = -2$ seconds. We ignore the $t = -2$ since it does not describe the physical situation.
   The binoculars will be travelling at $v(4)$ feet per second when they hit the ground. $s(t) = -16t^2 + 32t + 128$
   $v(t) = s'(t) = -32t + 32$
   $v(4) = (-32)(4) + 32 = -96$ feet per second.

6. [14 points] In the figure the straight line is tangent to the parabola at the point with x - coordinate $3/2$ and the parabola has equation $y = 3x^2 - 12x + 9$. Find the y - intercept, b, where the tangent line crosses the y - axis.

   (a) The y coordinate of the point of tangency is $3\,(3/2)^2 - 12\,(3/2) + 9 = -\frac{9}{4}$ so the point $\left(\frac{3}{2}, -\frac{9}{4}\right)$ is on the tangent line. Since $\frac{dy}{dx} = 6x - 12$, the slope of the tangent line at the point $\left(\frac{3}{2}, -\frac{9}{4}\right)$ is $m = 6\,(3/2) - 12 = -3$. Thus an equation for the tangent line is $y - \left(-\frac{9}{4}\right) = -3\left(x - \frac{3}{2}\right)$. Solving for y we get $y = -3x + \frac{9}{4}$. The y - intercept of the tangent line is $\frac{9}{4}$.

7. [14 points] The tangent line to the curve $y = f(x) = \frac{1}{3}x^3 - 2x^2 - 18x + 22$ is parallel to the line $6x - 2y = 1$ at two points on the curve. Find the two points.

   (a) The tangent line to the curve at the point $(a, f(a))$ has slope $f'(a) = a^2 - 4a - 18$. The line $6x - 2y = 1$ can be written $y = 3x - \frac{1}{2}$ so it has slope $m = 3$. Any line parallel to this line will have the same slope so we are looking for the numbers a that satisfy $a^2 - 4a - 18 = 3$ which can be written $a^2 - 4a - 21 = 0$. This quadratic can be factored into $(a-7)(a+3) = 0$ so we see that when $a = 7$ or $a = -3$, then the points $(7, f(7))$ and $(-3, f(-3))$ are where the tangent lines to $y = f(x)$ are parallel to the line $6x - 2y = 1$. Those two points are $\left(7, -\frac{263}{3}\right)$ and $(-3, -41)$.

   (b) $\frac{1}{3}(7)^3 - 2(7)^2 - 18(7) + 22 = -\frac{263}{3}$

   (c) $\frac{1}{3}(3)^3 - 2(3)^2 - 18(3) + 22 = -41$

8. [14 points] If you deposit \$100 into a savings account at the end of each month for two years, the balance will be a function $f(r)$ of the interest rate, $r\%$, At 7% interest (compounded monthly), $f(7) = 2568.10$ and $f'(7) = 25.06$. Approximately how much additional money would you earn if the bank paid $7\frac{1}{2}\%$ interest? [Hint: Use the formula for approximating the change in a function.]

   (a) We use the approximation $\frac{f(a+h)-f(h)}{h} \approx f'(a)$ or after simplifying, $f(a+h) - f(h) \approx f'(a) \cdot h$. Since the additional money earned is $f(7.5) - f(7)$ we have $f(7+0.5) - f(7) \approx f'(7) \cdot (0.5) = (25.06)(0.5) = \$12.53$. So we would earn an additional \$6.26 during those two years.