Mathematics 121-A Exam 2 Spring 2006

February 23, 2006

____
Name

Directions:

- **Only write on one side of each page.**
- **Use terminology correctly.**
- **Show your work: answers that can be obtained from a calculator will not receive credit.**
- **Partial credit is awarded for correct approaches so justify your steps.**

Do any seven (7) of the following.

Do not use a calculator to justify any problem except number 2.

1. [10 points] Use an ε, δ proof to show that $\lim_{x\to 4}(-2x+1) = -7$.

   (a) Let ε be any positive number and choose $\delta = \frac{1}{2}\varepsilon$. Then whenever $0 < |x-4| < \delta$ we have

$$\begin{aligned} |x-4| &< \delta \\ |x-4| &< \frac{1}{2}\varepsilon \\ |-2|\,|x-4| &< \varepsilon \\ |-2x+8| &< \varepsilon \\ |(-2x+1)-(-7)| &< \varepsilon \end{aligned}$$

2. [10 points] Given the limits $\lim_{x\to 1^+} \frac{2}{x-1}$, $\lim_{x\to 1^+} \frac{x^2-2x+1}{x-1}$, and $\lim_{x\to 6} \frac{\tan(\pi/x)}{x-1}$

   (a) In your own words, explain why the first limit does not exist but the other two do.

      i. The first limit does not exist because as x approaches 1 from the positive side the fraction $\frac{2}{x-1}$yields larger and larger values and in fact is unbounded above.

      ii. The second limit exists even though it has the form "$\frac{0}{0}$" because factoring the numerator allows us to isolate and remove the seeming divide by zero. See part b below for details.

      iii. The third limit exists because both the tangent function and $x-1$ are continuous on their domains. This means $\frac{\tan(\pi/x)}{x-1}$is continuous everywhere except at $x=0$ and $x=1$. Since the limit is as $x \to 6$, continuity tells us $\lim_{x\to 6} \frac{\tan(\pi/x)}{x-1} = \frac{\tan(\pi/6)}{6-1} = \frac{1}{15}\sqrt{3} \approx 0.115\,470$.

   (b) Evaluate the last two limits.

      i. $\lim_{x\to 1^+} \frac{x^2-2x+1}{x-1} = \lim_{x\to 1^+} \frac{(x-1)}{(x-1)} \cdot (x-1) = 1 \cdot 0 = 0.$

      ii. As explained in part (a) $\lim_{x\to 6} \frac{\tan(\pi/x)}{x-1} = \frac{\tan(\pi/6)}{6-1} = \frac{1}{15}\sqrt{3} \approx 0.115\,470$.

3. [15 points] Use the definition of continuity to determine if the function $f(x) = \begin{cases} \frac{x^2-9}{x-3}, & \text{if } x < 3 \\ 6, & \text{if } x = 3 \\ 5x-9, & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$.

   (a) The number 3 is in the domain of f and $f(3) = 6$ (the middle part of the function's definition)

(b) The function was changed in class so that "$x+3$" became "$x-3$".
$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(5x-9) = 15-9 = 6$ by continuity of polynomials.
$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x^2-9}{x-3} = \lim_{x\to 3^-} \frac{(x-3)}{(x-3)} \cdot (x+3) = 1 \cdot 6 = 6$
Thus $\lim_{x\to 3} f(x) = 6$ (it exists)

(c) Since $\lim_{x\to 3} f(x) = f(3)$ the function f is continuous at $x=3$.

4. [15 points] Do all of the following:

(a) Simplify $\log_2(16)\log_3\left(\frac{1}{27}\right)$

i. $\log_2(16)\log_3\left(\frac{1}{27}\right) = \log_2(2^4)\log_3(3^{-3}) = 4(-3) = -12$.

(b) Find the numbers x that solve the equation $\frac{e^{x^2}}{e^{x+6}} = 1$

i. $e^{x^2} = e^{x+6}$ so $x^2 = x+6$ giving $x^2-x-6=0$. Factoring we have $(x-3)(x+2)=0$ and we see $x = 3, -2$ solve the equation.

(c) If $\log_{\sqrt{b}}(106) = 2$ what is $\sqrt{b-25}$?

i. $\left(\sqrt{b}\right)^2 = 106$ which tells us that $b = 106$ so $\sqrt{b-25} = \sqrt{81} = 9$.

5. [15 points] Compute the derivative of $f(x) = \frac{x}{x+3}$ by evaluating the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

(a)

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\frac{x+h}{x+h+3} - \frac{x}{x+3}}{h} = \lim_{h\to 0} \frac{\frac{(x+h)(x+3)-x(x+h+3)}{(x+h+3)(x+3)}}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{3h}{(x+h+3)(x+3)} \\
&= \lim_{h\to 0} \frac{h}{h} \cdot \frac{3}{(x+h+3)(x+3)} \\
&= 1 \cdot \frac{3}{(x+3)^2}.
\end{aligned}$$

6. [20 points] Use the derivative rules for the following.

(a) Find $f'(x)$ if $f(x) = 3x^4 - 7x^2 + \frac{2}{x} + \sqrt{x}$.

i. $f(x) = 3x^4 - 7x^2 + 2x^{-1} + x^{1/2}$ so $f'(x) = 12x^3 - 14x^2 - 2x^{-2} + \frac{1}{2}x^{-1/2}$.

(b) If $h(x) = (x^3+x^2+1)(3x^2-4)$ use the product rule to find $h'(x)$.

i. $h'(x) = (3x^2+2x)(3x^2-4) + (x^3+x^2+1)(6x) = 15x^4 - 12x^2 + 12x^3 - 2x$ (for those of you who simplified)

(c) Find $\frac{dy}{dx}$ if $y = \frac{x^3+x}{2x^2-1}$

i. $\frac{dy}{dx} = \frac{(3x^2+1)(2x^2-1)-(x^3+x)(4x)}{(2x^2-1)^2}$ This was far enough but for those of you who simplified, $\frac{dy}{dx} = \frac{2x^4-5x^2-1}{(2x^2-1)^2}$.

(d) Find $\frac{d^3y}{dt^3}$ where $y = 2t^4 - 3t^3 + 4t - 6$. We have $\frac{dy}{dx} = 8t^3 - 9t^2 + 4$, $\frac{d^2y}{dt^2} = 24t^2 - 18t$ and $\frac{d^3y}{dt^3} = 48t - 18$.

7. [15 points] Do **one** (1) of the following

(a) Does the function $f(x) = x^3 + 2x^2 - 3x$ satisfy the equation $y''' + y'' + y' = 3x^2 + 10x + 7$?

i. Since

$$\begin{aligned} y &= x^3 + 2x^2 - 3x \\ y' &= 3x^2 + 4x - 3 \\ y'' &= 6x + 4 \\ y''' &= 6 \end{aligned}$$

adding the last three gives $(3x^2 + 4x - 3) + (6x + 4) + (6) = 3x^2 + 10x + 7$. So yes, the equation is satisfied.

(b) Find an equation for a tangent line to the graph of $f(x) = \frac{3x+5}{1+x}$ that is perpendicular to the line $2x - y = 1$. [There are two.]

i. $f(x) = \frac{3x+5}{1+x}$ so by the quotient rule $f'(x) = \frac{3(1+x)-(3x+5)(1)}{(1+x)^2} = \frac{-2}{(1+x)^2}$. The line our tangent line will be perpendicular to is $y = 2x - 1$ so our tangent line needs to have slope $-\frac{1}{2}$. We find the values of x where $f'(x) = -\frac{1}{2}$ by solving

$$\begin{aligned} \frac{-2}{(1+x)^2} &= -\frac{1}{2} \\ 4 &= (1+x)^2 \\ x^2 + 2x - 3 &= 0 \\ (x+3)(x-1) &= 0 \\ x &= 1, -3 \end{aligned}$$

For $x = 1$, $f(1) = \frac{8}{2} = 4$ and the tangent line is $y - 4 = -\frac{1}{2}(x - 1)$.
For $x = -3$, $f(-3) = \frac{-4}{-2} = 2$ and the tangent line is $y - 2 = -\frac{1}{2}(x + 3)$.