Mathematics 434 **Spring 2002** **Problem Sheet 6**

February 28, 2002

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Name

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. **Only write on one side of each page.**

"Anyone who cannot cope with mathematics is not fully human. At best he is a tolerable subhuman who has learned to wear shoes, bathe, and not make messes in the house." – Robert Heinlein in *Time Enough for Love.*

1 Problems

1. In class, we outlined a process (using the substitution principal) that shows, for any ring R, $R[x,y] \approx R[x][y]$. Fill in the details of that process or fill in the details of the following alternative.

   (a) Extend $R \to R[x][y]$ to a map $\Phi : R[x,y] \to R[x][y]$

   (b) Extend $R[x] \to R[x,y]$ to a map $\Psi : R[x][y] \to R[x,y]$

   (c) Use uniqueness of extension to show $\Phi\Psi$ and $\Psi\Phi$ are both the identity maps. (This shows Φ is an isomorphism).

2. Do **all** of the following

   (a) For which integers n does $x^2 + x + 1$ divide $x^4 + 3x^3 + x^2 + 6x + 10$ in $(\mathbf{Z}/n\mathbf{Z})[x]$?

   (b) Describe the kernel of the map defined by $\phi : \mathbf{Z}[x] \to \mathbf{R}$ given by $\phi(f(x)) = f\left(1 + \sqrt{2}\right)$.

3. Prove

   (a) the kernel of the homomorphism $\phi : \mathbf{C}[x,y] \to \mathbf{C}[t]$ given by $\phi(f(x,y)) = f(t^2, t^3)$ is the principal ideal generated by the polynomial $y^2 - x^3$.

   (b) describe the image of ϕ explicitly.

4. Let I, J be ideals of a ring R.

   (a) Show by example that $I \cup J$ need not be an ideal but show the set $I + J = \{r \in R : r = x + y,\ x \in I,\ y \in J$ is an ideal. This ideal is called the **sum** of I and J.

   (b) Prove that $I \cap J$ is an ideal.

   (c) Show by example that the set of products $\{xy : x \in I,\ y \in J\}$ need not be an ideal but that the set of finite sums $\sum_{i,j} x_i y_j$ of products of elements of I and J is an ideal. This ideal is called the **product** ideal and is denoted IJ.

   (d) Prove $IJ \subset I \cap J$.

   (e) Show by example that IJ and $I \cap J$ need not be equal.