# Double Integral Example Worksheet

## Double Integrals over general regions in $x, y$ coordinates

## Sketch regions too

1. $\int_0^4 \int_0^{4-x} xy\,dy\,dx$

   Inner: $\int_0^{4-x} xy\,dy = \frac{1}{2}xy^2\Big|_{y=0}^{4-x} = \frac{1}{2}x\left(-4+x\right)^2$

   Completion: $\int_0^4 \frac{1}{2}x\left(-4+x\right)^2 dx = \frac{1}{2}\int_0^4 \left(x^3 - 8x^2 + 16x\right)\ dx = \frac{32}{3}$

2. $\iint_D (x+y)\ dA$ where $D$ is the triangle with vertices $(0,0), (0,2), (1,2)$

   $\int_0^1 \int_{y=2x}^2 (x+y)\,dydx = \int_0^2 \int_0^{\frac{1}{2}y} (x+y)\,dxdy = \frac{5}{3}$.

3. $\iint_D 48xy\ dA$ where $D$ is the region bounded by $y = x^3$ and $y = \sqrt{x}$

   $\int_0^1 \int_{y=x^3}^{\sqrt{x}} (48xy)\ dy\ dx = \int_0^1 \int_{x=y^2}^{y^{\frac{1}{3}}} (48xy)\ dx\ dy = 5$

## Reverse order of integration.

1. $\int_0^1 \int_x^{2x} e^{y-x}\ dy\,dx = \int_{y=0}^1 \int_{x=\frac{1}{2}y}^{y} e^{y-x}\ dx\ dy + \int_1^2 \int_{\frac{1}{2}y}^{1} e^{y-x}\ dx\ dy$

2. $\int_0^{2\sqrt{3}} \int_{y^2/6}^{\sqrt{16-y^2}} 1\ dx\,dy = \int_0^2 \int_0^{\sqrt{6x}} 1\ dy\ dx + \int_2^4 \int_0^{\sqrt{16-x^2}} 1\ dy\ dx = \frac{2}{3}\sqrt{3} + \frac{8}{3}\pi$

3. $\int_0^7 \int_{x^2-6x}^{x} f(x,y)\ \ dy\ \ dx = \int_{-9}^0 \int_{3-\sqrt{y+9}}^{3+\sqrt{y+9}} f(x,y)\ dx\ dy + \int_0^7 \int_y^{3+\sqrt{y+9}} f(x,y)\ dx\ dy$

4. $\int_1^2 \int_x^{x^3} f(x,y)\ dy\,dx + \int_2^8 \int_x^8 f(x,y)\ dy\,dx = \int_1^8 \int_{y^{\frac{1}{3}}}^{y} f(x,y)\ dx\ dy$

## Find Volume of solid

1. Tetrahedron in first octant bounded by coordinate planes and $z = 7 - 3x - 2y$.

   $\int_0^{\frac{7}{3}} \int_0^{-\frac{3}{2}x+\frac{7}{2}} (7-3x-2y)\ dy\ dx = \int_0^{\frac{7}{3}} \int_0^{7-3x} \left(\frac{7-3x-z}{2}\right)\ dz\ dx = \int_0^{\frac{7}{2}} \int_0^{7-2y} \left(\frac{7-2y-z}{3}\right)\ dz\ dy = \frac{343}{36}$

2. Solid inside both the sphere $x^2+y^2+z^2 = 3$ and paraboloid $2z = x^2+y^2$.

   $\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \left(\sqrt{3-x^2-y^2} - \frac{x^2+y^2}{2}\right)\ dy\ dx = 2\sqrt{3}\pi - \frac{5}{3}\pi$

## Double Integrals using polar coordinates

## Direct Computations in polar coordinates

1. Compute $\int_0^{\pi/2} \int_1^3 re^{-r^2}\,dr\,d\theta$

   Inner: $\int_1^3 re^{-r^2}\,dr = -\frac{1}{2}e^{-9} + \frac{1}{2}e^{-1}$ Using $u = -r^2$ and $du = -2r\ dr$

   Completion: $\int_0^{\pi/2} \int_1^3 re^{-r^2}\,dr\,d\theta = -\frac{1}{4}e^{-9}\pi + \frac{1}{4}e^{-1}\pi$

2. FInd the area bounded by the cardioid $r = 1 + \sin\theta$.

$$\iint\limits_D 1\ dA = \int_0^{2\pi} \int_0^{1+\sin(\theta)} 1\ r\ dr\ d\theta = \frac{3}{2}\pi$$

3. Find the area bounded by one leaf of the rose $r = 4\cos\theta$

$$\iint\limits_D 1\ dA = \int_0^{\pi} \int_0^{4\cos(\theta)} 1\ r\ dr\ d\theta = 4\pi$$

4. Find area inside both $r = 1$ and $r = 2\sin\theta$.

$$\iint\limits_D 1\ dA = 2\left(\int_0^{\frac{\pi}{6}} \int_0^{2\sin(\theta)} 1\ r\ dr\ d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_0^{1} 1\ r\ dr\ d\theta\right) = \frac{2}{3}\pi - \frac{1}{2}\sqrt{3}$$

**Convert from Cartesian ( $x, y$) to polar coordinates before integrating**

1. Find $\iint_D f(x,y)\ dA$ where $D$ is the region bounded by the $x$-axis, the line $y = x$ and the circle $x^2 + y^2 = 1$.

   $\iint_D f(x,y)\ \ dA = \int_0^{\frac{\pi}{4}} \int_0^1 f(r\cos(\theta), r\sin(\theta))\ \ r\ dr\ d\theta$

2. Find the volume of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the $xy$-plane.

   $V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - y^2)\ \ dy\ dx = \int_0^{2\pi} \int_0^2 (4 - r^2)\ \ r\ dr\ d\theta = 8\pi$

3. Find the volume inside the sphere $x^2 + y^2 + z^2 = 25$ and outside the cylinder $x^2 + y^2 = 9$.

   $V = \iint\limits_D \left(\sqrt{25 - x^2 - y^2} - 0\right)\ dA = 2\int_0^{2\pi} \int_3^5 \sqrt{25 - r^2}\ r\ dr\ d\theta = \frac{256}{3}\pi$

4. Find the volume inside the sphere $x^2 + y^2 + z^2 = 25$ and outside the cylinder $(x-1)^2 + y^2 = 1$.

   [**This is a project problem but a hint is to write the equation of the cylinder in polar coordinates.**]