

# Double Integral Example Worksheet

## Double Integrals over general regions in $x, y$ coordinates

### Sketch regions too

1.  $\int_0^4 \int_0^{4-x} xy \, dy \, dx$

Inner:  $\int_0^{4-x} xy \, dy = \frac{1}{2}xy^2 \Big|_{y=0}^{4-x} = \frac{1}{2}x(-4+x)^2$

Completion:  $\int_0^4 \frac{1}{2}x(-4+x)^2 \, dx = \frac{1}{2} \int_0^4 (x^3 - 8x^2 + 16x) \, dx = \frac{32}{3}$

2.  $\iint_D (x+y) \, dA$  where  $D$  is the triangle with vertices  $(0,0)$ ,  $(0,2)$ ,  $(1,2)$

$\int_0^1 \int_{y=2x}^2 (x+y) \, dy \, dx = \int_0^1 \int_0^{\frac{1}{2}y} (x+y) \, dx \, dy = \frac{5}{3}$ .

3.  $\iint_D 48xy \, dA$  where  $D$  is the region bounded by  $y = x^3$  and  $y = \sqrt{x}$

$\int_0^1 \int_{y=x^3}^{\sqrt{x}} (48xy) \, dy \, dx = \int_0^1 \int_{x=y^2}^{\frac{1}{3}} (48xy) \, dx \, dy = 5$

### Reverse order of integration.

1.  $\int_0^1 \int_x^{2x} e^{y-x} \, dy \, dx = \int_{y=0}^1 \int_{x=\frac{1}{2}y}^y e^{y-x} \, dx \, dy + \int_1^2 \int_{\frac{1}{2}y}^1 e^{y-x} \, dx \, dy$

2.  $\int_0^{2\sqrt{3}} \int_{y^2/6}^{\sqrt{16-y^2}} 1 \, dx \, dy = \int_0^2 \int_0^{\sqrt{6x}} 1 \, dy \, dx + \int_2^4 \int_0^{\sqrt{16-x^2}} 1 \, dy \, dx = \frac{2}{3}\sqrt{3} + \frac{8}{3}\pi$

3.  $\int_0^7 \int_{x^2-6x}^x f(x,y) \, dy \, dx = \int_{-9}^0 \int_{3-\sqrt{y+9}}^{3+\sqrt{y+9}} f(x,y) \, dx \, dy + \int_0^7 \int_y^{3+\sqrt{y+9}} f(x,y) \, dx \, dy$

4.  $\int_1^2 \int_x^{x^3} f(x,y) \, dy \, dx + \int_2^8 \int_x^8 f(x,y) \, dy \, dx = \int_1^8 \int_{\frac{1}{y^3}}^y f(x,y) \, dx \, dy$

### Find Volume of solid

1. Tetrahedron in first octant bounded by coordinate planes and  $z = 7 - 3x - 2y$ .

$\int_0^{\frac{7}{3}} \int_0^{-\frac{3}{2}x+\frac{7}{2}} (7-3x-2y) \, dy \, dx = \int_0^{\frac{7}{3}} \int_0^{7-3x} \left(\frac{7-3x-z}{2}\right) \, dz \, dx = \int_0^{\frac{7}{2}} \int_0^{7-2y} \left(\frac{7-2y-z}{3}\right) \, dz \, dy = \frac{343}{36}$

2. Solid inside both the sphere  $x^2 + y^2 + z^2 = 3$  and paraboloid  $2z = x^2 + y^2$ .

$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \left(\sqrt{3-x^2-y^2} - \frac{x^2+y^2}{2}\right) \, dy \, dx = 2\sqrt{3}\pi - \frac{5}{3}\pi$

### Double Integrals using polar coordinates

#### Direct Computations in polar coordinates

1. Compute  $\int_0^{\pi/2} \int_1^3 re^{-r^2} \, dr \, d\theta$

Inner:  $\int_1^3 re^{-r^2} \, dr = -\frac{1}{2}e^{-9} + \frac{1}{2}e^{-1}$  Using  $u = -r^2$  and  $du = -2r \, dr$

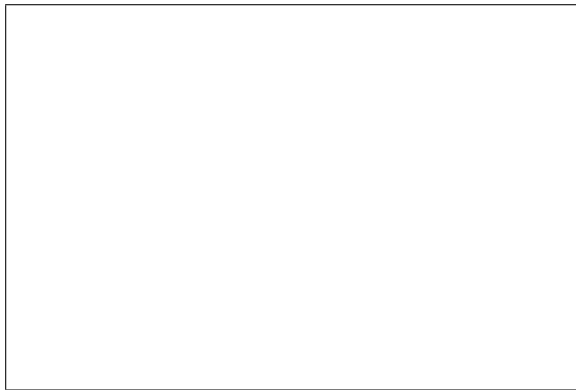
Completion:  $\int_0^{\pi/2} \int_1^3 re^{-r^2} \, dr \, d\theta = -\frac{1}{4}e^{-9}\pi + \frac{1}{4}e^{-1}\pi$

2. Find the area bounded by the cardioid  $r = 1 + \sin \theta$ .



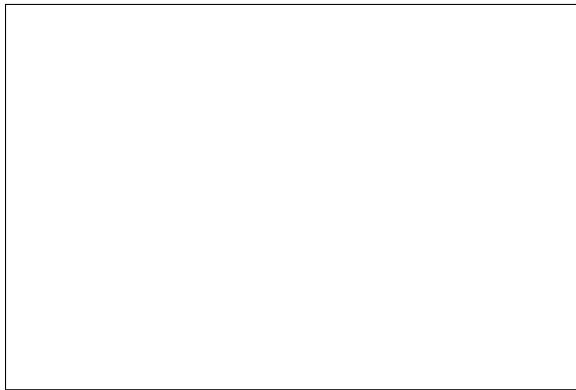
$$\iint_D 1 \, dA = \int_0^{2\pi} \int_0^{1+\sin(\theta)} 1 \, r \, dr \, d\theta = \frac{3}{2}\pi$$

3. Find the area bounded by one leaf of the rose  $r = 4 \cos \theta$



$$\iint_D 1 \, dA = \int_0^\pi \int_0^{4 \cos(\theta)} 1 \, r \, dr \, d\theta = 4\pi$$

4. Find area inside both  $r = 1$  and  $r = 2 \sin \theta$ .



$$\iint_D 1 \, dA = 2 \left( \int_0^{\frac{\pi}{6}} \int_0^{2 \sin(\theta)} 1 \, r \, dr \, d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_0^1 1 \, r \, dr \, d\theta \right) = \frac{2}{3}\pi - \frac{1}{2}\sqrt{3}$$

**Convert from Cartesian (  $x, y$  ) to polar coordinates before integrating**

1. Find  $\iint_D f(x, y) dA$  where  $D$  is the region bounded by the  $x$ -axis, the line  $y = x$  and the circle  $x^2 + y^2 = 1$ .

$$\iint_D f(x, y) dA = \int_0^{\frac{\pi}{4}} \int_0^1 f(r \cos(\theta), r \sin(\theta)) r dr d\theta$$

2. Find the volume of the solid bounded by the paraboloid  $z = 4 - x^2 - y^2$  and the  $xy$ -plane.

$$V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - y^2) dy dx = \int_0^{2\pi} \int_0^2 (4 - r^2) r dr d\theta = 8\pi$$

3. Find the volume inside the sphere  $x^2 + y^2 + z^2 = 25$  and outside the cylinder  $x^2 + y^2 = 9$ .

$$V = \iint_D (\sqrt{25 - x^2 - y^2} - 0) dA = 2 \int_0^{2\pi} \int_3^5 \sqrt{25 - r^2} r dr d\theta = \frac{256}{3}\pi$$

4. Find the volume inside the sphere  $x^2 + y^2 + z^2 = 25$  and outside the cylinder  $(x - 1)^2 + y^2 = 1$ .

[This is a project problem but a hint is to write the equation of the cylinder in polar coordinates.]