1 Mathematics 434-A Spring 2001

1.1 Exam 1

March 26, 2001

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Name

Textbook/Notes used: ______________________________

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use a text or notes or technology. **Only write on one side of each page.**

Do any five (5) of the following.

1. Analyze the ring obtained from the integers $\mathbf{Z}$ by adjoining an element α which satisfies both of the relations $\alpha^3 + \alpha^2 + 1 = 0$ and $\alpha^2 + \alpha = 0$.In particular, what is the form of an arbitrary element in the ring $\mathbf{Z}[\alpha]$? [Hint: linear combination consequences of the relations give a lot of compression.]
2. Let $\phi : \mathbf{Z}[x] \to \mathbf{R}$ be the ring homomorphism given by $\phi(f(x)) = f\left(1+\sqrt{2}\right)$. Show that $\ker(\phi) = (x^2 - 2x - 1)$ the ideal generated by $x^2 - 2x - 1$.
3. Let R be a ring and a_1, a_2, a_3 elements of R. The ideal $I = (a_1, a_2, a_3)$ generated by a_1, a_2, a_3 is defined to be the smallest ideal of R containing a_1, a_2, a_3. Prove that I is just the set of all linear combinations $J = \{r_1 a_1 + r_2 a_2 + r_3 a_3 : r_1, r_2, r_3 \in R\}$. You may use the fact that J is an ideal of R without proving it.
4. Find the signature of the quadratic form associated with the conic $x^2 + 2xy + y^2 = 1$.

   Hint: The eigenvalues of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ are $2, 0$ with respective eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
5. Let V be a Euclidean Space. If $v, w \in V$ have the property that $|v| = |w|$, show $(v + w) \perp (v - w)$.
6. Let W, W_1, W_2 be subspaces of a vector space V with a symmetric bilinear form. Prove if $W_1 \subset W_2$ then $W_2^{\perp} \subset W_1^{\perp}$.
7. Extend the vector $X_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to an orthonormal basis for R^3 (with the usual dot product).
8. Use the Sylow theorems to show no group of order $98 = 2 \cdot 7^2$ can be simple.