Mathematics 434 Spring 2001 Problem Sheet 3

February 5, 2001

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Name

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. **Only write on one side of each page.**
"Mathematics seems to endow one with something like a new sense." – Charles Darwin

1 Problems

1.1 Geometry Associated with Real Positive Definite Form

1. Let W be a subspace of a Euclidean space V. Prove $W = W^{\perp\perp}$.

2. Find the matrix of the projection $T : R^3 \rightarrow R^2$ such that the image of the standard basis of R^3 forms an equilateral triangle and $T(e_1)$ points in the direction of the x - axis.

3. Let $w \in R^n$ be a vector of unit length.

   (a) Prove the matrix $P = I - 2ww^t$ is orthogonal.

   (b) Prove multiplication by P is a reflection throught the space $W^{\perp}$ orthogonal to w. That is, prove if we write an arbitrary vector $v = cw + w'$ where $w' \in W^{\perp}$, then $Pv = -cw + w'$.

   (c) Let X, Y be arbitrary vectors in R^n with the same length. Determine a vector w such that $PX = Y$. [Hint: draw generic $X + Y$ and $X - Y$].

4. Use the above problem (number 3) to prove every orthogonal $n \times n$ matrix is a product of at most n reflections.

1.2 Hermitian Forms

1. Prove a matrix A is hermitian if and only if the associated form X^*AX is a hermitian form.

2. Is $\langle X, Y \rangle = x_1y_1 + ix_1y_2 - ix_2y_1 + ix_2y_2$ on C^2 a hermitian form?

3. Prove the determinant of a hermitian matrix is a real number.

4. Let P_n be the vector space of polynomials of degree less than or equal to n.

   (a) Show
   $$\langle f, g \rangle = \int_0^{2\theta} \overline{f\left(e^{i\theta}\right)} g\left(e^{i\theta}\right) \, d\theta$$
   is a positive definite hermitian form on P_n.

   (b) Find an orthonormal basis for this form when $n = 3$.

5. Determine whether or not the following rules define hermitian forms on the space $C^{n \times n}$ of complex matrices and, if so, determine their signature.

   (a) $\langle A, B \rangle = Trace\left(A^*B\right)$.

(b) $\langle A, B \rangle = Trace\left(\overline{A}B\right)$.

1.3 Spectral Theorem

1.

   (a) Find a unitary matrix P so that PAP^* is diagonal when

$$A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}.$$

   (b) Find a real orthogonal matrix P so that PAP^t is diagonal when

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

2. Prove a real symmetric matrix is positive definite if and only if all of its eigenvalues are positive.

1.4 Conics and Quadrics

1. Determine the type of the quadric

$$x^2 + 4xy + 2xz + z^2 + 3x + z - 6 = 0.$$

2.

   (a) Describe the types of conic in terms of the signature of the quadratic form.
   (b) Do the same for quadrics in R^3.