Mathematics 433-A Fall 2007

Exam 1

October 9, 2007

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Name

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use a text or notes or technology. Include a careful sketch of any graph obtained by technology in solving a problem. **Only write on one side of each page.**

The Problems

1. Prove that if $\phi : G \to G'$ is a group homomorphism and $g \in G$ has finite order then $|\phi(g)|$ divides $|g|$.

2. Do **one** (1) of the following.

   (a) If $\phi : G \to G'$ is a surjective group homomorphism and K is a normal subgroup of G, then we have already shown that $\phi(K) = \{\phi(k) : k \in K\}$ is a subgroup of G'. Prove that since K is normal in G, $\phi(K)$ must be normal in G'.

   (b) If $\phi : G \to G'$ is a group homomorphism and K' is a normal subgroup of G', then we have already shown that $\phi^{-1}(K') = \{k \in G : \phi(k) \in K'\}$ is a subgroup of G.Prove that since K' is normal in G' $\phi^{-1}(K')$ must be normal in G.

3. Let $G = \langle a \rangle$ be an infinite cyclic group. Find two automorphisms of G and show that there are no other automorphisms of G.

4. Given a group G, subgroup H of G and element $g \in G$,we define the **conjugate subgroup of** H in G to be the set
$$gHg^{-1} = \left\{ghg^{-1} : h \in H\right\}.$$
Prove gHg^{-1} is indeed a subgroup of G.

5. Do **one** of the following.

   (a) If G contains exactly one element of order 2,prove that element is in the center of G. [Hint: consider conjugates of that element.]

   (b) Let G be an abelian group of odd (and hence finite) order. Prove the map $\phi : G \to G$ defined by $\phi(x) = x^2$ is an automorphism.

6. Let $\phi : G \to G'$ be a group homomorphism with kernel K.Let H be another subgroup of G. Recall that $HK = \{hk : h \in H, k \in K\}$ .Show $\phi^{-1}(\phi(H)) = HK$.

Definitions you should know.

1. The **general linear** group of order n over the real numbers $GL(n, \mathbf{R})$.

2. The **center,** $Z(G)$, of a group G.

3. The **centralizer,** $C(a)$, of an element a in a group G.

4. A **normal** subgroup N of a group G.
5. A **homomorphism** from the group G to the group G'.
6. An **automorphism** of the group G.