Mathematics 433-A **Fall 2006**

Final Examination

December 13, 2006

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Name

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Directions: **Only write on one side of each page.**

Prove and six (6) of the following.

1. Let G be a group. The **commutator subgroup,** G', of G is the normal subgroup generated by the set $\{aba^{-1}b^{-1} : a, b \in G\}$ .That is, every element of G' is a product

$$x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}$$

   where each $x_i$has the form $x\left(aba^{-1}b^{-1}\right)x^{-1}$, each $i_j = \pm 1$, and k is any positive integer.

   If $a, b \in G$ then we define the **commutator** of a and b to be $[a, b] = aba^{-1}b^{-1}$. (Note that a and b commute if and only if their commutator $[a, b] = e$.)

   Prove that G' is a normal subgroup of G.

2. Show that there is no simple group of order pqr where p, q, r are distinct primes.

3. Suppose G is a finite group with $|G| = p^n q^m$ where p and q are distinct primes. Suppose further that all Sylow subgroups of G are normal. Show G is isomorphic to the product of its Sylow subgroups.

4. Prove that if $G/Z(G)$ is cyclic then G is abelian.

5. Show the center of a group of order 60 cannot have order 4 by considering $G/Z(G)$.

6. Let Q denote the group of rational numbers under addition. Let $\phi : Q \to Q$ be an arbitrary automorphism and suppose $\phi(1) = a$.

   (a) Prove $\phi(1/2) = a/2$.

   (b) Generalize the above to prove that if n is any integer $\phi(1/n) = a/n$.

   (c) Deduce $\phi(x) = x\phi(1)$ for all $x \in Q$.

7. What is the stabilizer of the coset aH for the action of G on $G/H = \{xH : x \in G\}$ where the action is left multiplication.

8. Let N be a normal subgroup of a group G. Suppose that $|N| = 5$ and that $|G|$ is an odd integer. Prove that N is contained in the center of G.

9. Classify all groups of order 18.

10. Let F be the free group on the alphabet $S = \{a_1, a_2, a_3, \cdots\}$, $R = \{r_1, r_2, \cdots, r_k\}$ be a collection of words (elements) of F, and N be the intersection of all normal subgroups of F that contain the set R. Prove that the set, T, of all finite products of conjugates of words in R contains N.