Mathematics 433-A **Fall 2006**

Exam 2

November 14, 2006

____
Name

____

Directions: Only write on one side of each page.

Do any six (6) of the following

1. Let $K \subset H \subset G$ be subgroups of a **finite** group G. Prove the formula

$$[G : K] = [G : H]\,[H : K]\,.$$

2. Do **both** of the following:

   (a) If S is a set and G is a group acting on S, prove that the relation

   $$s \sim s' \text{ if } s' = gs \text{ for some } g \in G$$

   is an equivalence relation.

   (b) Let $\phi : G \to G\,'$ be a homomorphism, and let S be a set on which $G\,'$ acts. Use ϕ to define, with proof, a group action of G on S.

3. Do one of the following

   (a) Let G be a group containing normal subgroups of orders 3 and 5, respectively. Prove G contains an element of order 15.

   (b) Let H, K be subgroups of a group G. Show the set of products $HK = \{hk : h \in H,\ k \in K\}$ is a subgroup if and only if $HK = KH$.

4. Do one of the following:

   When we classified the group M of rigid motions of the plane we claimed the following six relations were all true and proved a few of them.

   (a) Add to our certainty by **algebraically** proving either part iv. or part v.

      i. $t_a t_b = t_{a+b}$

      ii. $\rho_\theta \rho_\eta = \rho_{\theta+\eta}$

      iii. $rr = i$

      iv. $\rho_\theta t_a = t_{a'} \rho_\theta$, where $a' = \rho_\theta(a)$

      v. $r t_a = t'_a r$, where $a' = r(a)$

      vi. $r \rho_\theta = \rho_{-\theta} r$.

   (b) Use the above relations to show that if m is an orientation reversing motion of the plane then m^2 is a translation.

   (c) Compute the glide vector of the glide $t_{\vec{a}} \rho_\theta r$ in terms of $\vec{a}$ and θ.

Figure 1: 1 2

5. Do one of the following:

   (a) Let G be a group and $Aut\left(G\right)$ the group of automorphisms of G. Prove or disprove: The set of inner automorphisms $Inn\left(G\right) = \{\phi \in Aut\left(G\right) : \phi\left(g\right) = xgx^{-1} \text{ for some } x \in G\}$ is a normal subgroup of $Aut\left(G\right)$.

   (b) Let S be a set on which a group G operates. Let $H = \{g \in G : gs = s \text{ for all } s \in S\}$. Prove H is a normal subgroup of G.

6. The following patterns represent small portions of two tilings of the infinite plane. Circle one of the following patterns and let G be the group of symmetries of that tiling. Determine the point group of G.

7. Let G be a group acting on the set S. Let s be a fixed element in S and t an element in the orbit of s, say $t = as$. Prove the stabilizer of t in G is a conjugate subgroup of the stabilizer of s in G. Specifically, show $G_t = aG_sa^{-1}$.

8. Determine the group of automorphisms $Aut\left(G\right)$ if $G = C_2 \times C_2$.

Useful Facts

-
$$\rho_\theta\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \cos\left(\theta\right) & -\sin\left(\theta\right) \\ \sin\left(\theta\right) & \cos\left(\theta\right) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$r\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$