Mathematics 433-A — Fall 2006

Exam 1

September 26, 2006

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Name

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use a text or notes or technology. Include a careful sketch of any graph obtained by technology in solving a problem. **Only write on one side of each page.**

The Problems

1. (10, 10 points)
   (a) Use one of the principles of mathematical induction to prove if a, b, c are elements in a group G for which $b = cac^{-1}$, then $b^n = ca^n c^{-1}$ is true for all positive integers n.
   (b) Prove that $b^n = ca^n c^{-1}$ is also true for all **negative** integers n. Prove that if $\phi : G \to H$ is a group homomorphism then $\phi\left(a^{-1}\right) = \left(\phi(a)\right)^{-1}$ for every $a \in G$

2. (10, 10 points) Let G be a group and $\phi : G \to G$ be the map $\phi(x) = x^{-1}$.
   (a) Prove that ϕ is a bijection
   (b) Prove that ϕ is an automorphism if and only if G is abelian.

3. (15 points each) Do four (4) of the following problems.
   (a) Prove that every subgroup of a cyclic group is cyclic.
   (b) Prove that a group in which every element except the identity has order 2 is abelian.
   (c) Find all automorphisms of the group $(Z, +)$ of integers under the operation of addition. [Recall that every subgroup of $(Z, +)$ has the form bZ.]
   (d) (15 points) Let ϕ, ψ be two homomorphisms from a group G to another group G' and let $H \subset G$ be the subset of G given by $H = \{x \in G : \phi(x) = \psi(x)\}$ .Prove or disprove, H is a subgroup of G.
   (e) Let H be a subgroup of a group G. Prove that the relation defined by the rule $a\tilde{}b$ if and only if $b^{-1}a \in H$ is an equivalence relation on G.
   (f) The orders of the elements in $U(20)$ and $U(24)$ are given in the tables below. Prove that these two groups are not isomorphic by proving that if $\phi : G \to H$ is an isomorphism, then the order of a must equal the order of $\phi(a)$, $|a| = |\phi(a)|$.

$U(20)$	1	3	7	9	11	13	17	19
Order	1	4	4	2	2	4	4	2

$U(24)$	1	5	7	11	13	17	19	23
Order	1	2	2	2	2	2	2	2

Definitions you should know.

1. The **general linear** group of order n over the real numbers $GL(n, \mathbf{R})$.
2. The **center,** $Z(G)$, of a group G.
3. The **centralizer,** $C(a)$, of an element a in a group G.
4. A **normal** subgroup N of a group G.
5. A **homomorphism** from the group G to the group G'.