1 Additional Exercises: Modular Arithmetic

1. Let G be the group of invertible, real, upper triangular 2×2 matrices. Determine whether the or not the following sets are normal subgroups H of G. If they are, use the first isomorphism theorem to identify G/H.

   (a) $H = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : a_{11} = 1 \right\}$

   (b) $H = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : a_{12} = 0 \right\}$

   (c) $H = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : a_{11} = a_{22} \right\}$

   (d) $H = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : a_{11} = a_{22} = 1 \right\}$

2. Identify the quotient group $\mathbf{R}^x/P$ where $\mathbf{R}^x$ is the group of all non-zero real numbers under the binary operation of multiplication and P denotes the subgroup of positive real numbers.

3. Find all normal subgroups N of the quaternion group H and identify the quotients H/N.

4. Prove the subset H of $G = GL(n, \mathbf{R})$ of matrices whose determinant is positive forms a normal subgroup, and describe the quotient group G/H.

5. Let $K \subset H \subset G$ be subgroups of a finite group G. Prove the formula

$$[G : K] = [G : H]\,[H : K]\,.$$