Final Exam (December 12)

I affirm this work abides by the university's Academic Honesty Policy.

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Print Name, then Sign

Directions:

- **Only write on one side of each page.**
- **Use terminology correctly.**
- **Partial credit is awarded for correct approaches so justify your steps.**

Do BOTH of the following.

R.1. Let V be the subset of $\mathbf{C}^\infty$ (see Useful Information below) defined by

$$V = \left\{ (a_1, a_2, a_3, \cdots) \in \mathbf{C}^\infty \;\middle|\; \begin{array}{c} a_1, a_2 \in \mathbf{C} \\ a_k = a_{k-1} + 6a_{k-2}, \quad k \geq 3 \ . \end{array} \right\}$$

Prove that V is a subspace of $\mathbf{C}^\infty$.

R.2. Use the method of Mathematical Induction to prove the following property of inner products is true for every positive integer n.

$$\text{If } \vec{x}_1, \cdots, \vec{x}_n, \vec{y} \in \mathbf{C}^m, \text{ then } \left\langle \sum_{k=1}^{n} \vec{x}_k, \ \vec{y} \right\rangle = \sum_{k=1}^{n} \langle \vec{x}_k, \ \vec{y} \rangle \ .$$

Do TWO (2) of these "In Class, Text, or Homework" problems

M.1. [15 points] A certain 6×6 matrix C can be written as $C = AB$ where A is 6×4 and B is 4×6. Explain how you know that $\det(C) = 0$.

M.2. [15 points] Prove **Theorem FTMR**, Fundamental Theorem of Matrix Representation:

1. Suppose that $T : U \to V$ is a linear transformation, $B = \{\vec{u}_1, \cdots, \vec{u}_n\}$ is a basis for U, $C = \{\vec{v}_1, \cdots, \vec{v}_m\}$ is a basis for V and $M^T_{B,C}$ is the matrix representation of T relative to B and C. Then, for any $\vec{u} \in U$, $\rho_C(T(u)) = M^T_{B,C}(\rho_B(\vec{u}))$

M.3. [15 points] Prove: If A is diagonalizable, then A^T is similar to A.

M.4. [15 points] Prove**Theorem CLTLT**, Composition of Linear Transformations is a Linear Transformation using the definition of linear transformation.

1. Suppose that $T : U \longmapsto V$ and $S : V \longmapsto W$ are linear transformations. Then $(S \circ T) : U \longmapsto W$ is a linear transformation.

Do any THREE (3) of these "Other" problems

T.1. [5, 10 points] Suppose that $T : V \longrightarrow V$ is a linear transformation. Prove that $(T \circ T)(\vec{v}) = \vec{0}$ for every $\vec{v} \in V$ if and only if $R(T) \subseteq K(T)$ (the range of is a subset of the kernel of T).

T.2. [15 points] Determine if $T : P_2 \to P_2$ defined by $T(f(x)) = f(x+1)$ is an isomorphism. You may assume T has already been shown to be a linear transformation.

T.3. [15 points] We say B is a **square root** of A if A and B are both square and $B^2 = A$.Clearly, the diagonal matrix $D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$ has $\begin{bmatrix} (d_{11})^{1/2} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (d_{nn})^{1/2} \end{bmatrix}$ as a square root.

1. Prove that if A is a **diagonalizable** matrix, then A has a square root.

T.4. [15 points] Given $A \in M_{m,n}$ and $B \in M_{n,m}$, show that if $N(BA - I_n) = \left\{\vec{0}_n\right\}$ then $N(AB - I_m) = \left\{\vec{0}_m\right\}$. [Here, $N(D)$ refers to the null space of D. Be careful! Neither A nor B is square.]

Do any TWO (2) of these "Computational" problems

C.1. [15 points] Let V be the vector space of all 3×3 upper-triangular matrices with trace 0 and let $T : V \longrightarrow \mathbf{C}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & -a-d \end{bmatrix}\right) = \begin{bmatrix} a+e \\ b-2c+d \\ 2a+3d \end{bmatrix}$$

1. Find a basis for the kernel of T, $\ker(T)$.

C.2. [15 points] Given the matrix $A = \begin{bmatrix} 1 & 0 & -3 & 1 \\ 2 & 1 & -7 & 4 \end{bmatrix}$. Find a 4×4 matrix B for which the null space of A is the same as the column space of B. That is, find B so that $N(A) = C(B)$..

C.3. [15 points] The matrix $M^T_{B,C}$ given below is the matrix representation of a certain linear transformation $T : P_3 \longrightarrow P_2$ with respect to the "standard" bases $B = \left\{1, x, x^2, x^3\right\}$ for P_3 and $C = \left\{1, x, x^2\right\}$ for P_2.

$$M^T_{B,C} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & -1 \end{bmatrix}.$$

Compute $T\left(2 + 3x - 4x^2 + 5x^3\right)$.

Useful Information

- The set of all infinite sequences with complex numbers as terms, $\mathbf{C}^\infty = \{(a_1, a_2, a_3, \cdots) \mid a_i \in \mathbf{C},\ 1 \le i\}$ is a vector space under the following definitions of equality, addition, and scalar multiplication.

  1. $(a_1, a_2, a_3, \cdots) = (b_1, b_2, b_3, \cdots)$ if and only if $a_i = b_i$ for each $i \ge 1$.
  2. $(a_1, a_2, a_3, \cdots) + (b_1, b_2, b_3, \cdots) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \cdots)$
  3. $\alpha(a_1, a_2, a_3, \cdots) = (\alpha a_1, \alpha a_2, \alpha a_3, \cdots)$