Final Exam (December 15)

I affirm this work abides by the university's Academic Honesty Policy.

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Print Name, then Sign

Directions:

- **Only write on one side of each page.**
- **Use terminology correctly.**
- **Partial credit is awarded for correct approaches so justify your steps.**

Do any TWO (2) of these "Computational" problems

C.1. [15 points] The sets $B = \{1+x,\ \ 2+x^2,\ \ 3+x+x^2\}$ and $D = \{3,\ \ 2-x,\ \ 1-x^2\}$ are both bases for P_2.Compute the change of basis matrix $C_{B,D}$ and use it to compute $\rho_D\left(5(1+x)+4\left(2+x^2\right)\right)$

C.2. [15 points] Using anything you know about determinants, compute the determinant of the following matrix **by hand**

$$A = \begin{bmatrix} 0 & 2 & 2 & 3 & 0 \\ 0 & 4 & 4 & 6 & 1 \\ 1 & 2 & 3 & 6 & 0 \\ -1 & 4 & 0 & 3 & 0 \\ -1 & 2 & 3 & 6 & 0 \end{bmatrix}$$

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C.3. [8, 7 points] Given the matrix $A = \begin{bmatrix} 1 & 0 & -3 & 1 \\ 2 & 1 & -8 & 3 \end{bmatrix}$.

1. Find a 4×4 matrix B for which the null space of A is the same as the column space of B. That is, find B so that $N(A) = C(B)$.
2. Now find a matrix F so that $N(F) = C(A)$

Do any TWO (2) of these "In Class, Text, Homework, or Similar" problems

M.1. [15 points] Prove that a subset W of a vector space V is a subspace if and only if $\alpha\vec{w}_1 + \beta\vec{w}_2 \in W$ is true for all $\vec{w}_1, \vec{w}_2 \in W$ and for all $\alpha, \beta \in \mathbf{C}$.

M.2. [15 points] Prove Theorem FTMR, Fundamental Theorem of Matrix Representation:

1. Suppose that $T : U \to V$ is a linear transformation, $B = \{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n\}$ is a basis for U, C is a basis for V and $M^T_{B,C}$ is the matrix representation of T relative to B and C. Then, for any $\vec{u} \in U$, $\rho_C(T(u)) = M^T_{B,C}(\rho_B(\vec{u}))$.

M.3. [15 points] Let $T, U : \mathbf{C}^n \to \mathbf{C}^m$ be linear transformations. Prove the function $T + U : \mathbf{C}^n \to \mathbf{R}^m$ defined by $(T+U)(\overrightarrow{x}) = T(\overrightarrow{x}) + U(\overrightarrow{x})$ for all $\overrightarrow{x}$ in $\mathbf{C}$ is also a linear transformation.

M.4. [15 points] Prove that cancellation holds in a vector space. That is, prove the following theorem.

If V is a vector space, $\vec{v}_1, \vec{v}_2 \in V$, and $\alpha, \beta \in \mathbf{C}$. Then

1. If $\alpha\vec{v}_1 = \beta\vec{v}_1$ and $\vec{v}_1 \neq \vec{0}$ then $\alpha = \beta$ and
2. If $\alpha\vec{v}_1 = \alpha\vec{v}_2$ and $\alpha \neq 0$ then $\vec{v}_1 = \vec{v}_2$

Do any THREE (3) of these "Other" problems

T.1. [5, 10 points] Let V be a vector space and $Z = \left\{\vec{0}_V\right\}$. Define a function $S_Z : V \to Z$ by $S_Z(\vec{v}) = \vec{0}_V$ for all $\vec{v} \in V$.

1. Prove that S_Z is a linear transformation. (You need not prove that Z is a vector space.)
2. Prove that $T : V \to V$ is surjective if and only if $K(S_Z) \subseteq R(T)$ (the kernel of S_Z is a subset of the range of T.)

$$V \xrightarrow{T} V \xrightarrow{S_Z} \vec{0}$$

T.2. [15 points] Prove that if a set $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \cdots, \vec{v}_n\}$ is a linearly dependent set of nonzero vectors, then there is an index t for which $\vec{v}_t$ is equal to a linear combination of the vectors $\vec{v}_{t+1}, \vec{v}_{t+2}, \cdots, \vec{v}_n$ that **follow** it in S.

T.3. [15 points] Given an $m \times n$ matrix A and an $n \times m$ matrix B where $m \neq n$. Show that if $N(BA - I_n) = \left\{\vec{0}_n\right\}$ then $N(AB - I_m) = \left\{\vec{0}_m\right\}$. [Here, $N(D)$ refers to the null space of D. Be careful! Neither A nor B is square.]

T.4. [15 points] Suppose A is a square matrix with the property that $\ker\left(A^2\right) = \ker\left(A^3\right)$. Prove that $\ker\left(A^3\right) = \ker\left(A^4\right)$.

You must do both of these problems ON THIS SHEET

R.1. [15 points] Prove that the set $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{C}^3 : 5x_1 - 7x_2 - 2x_3 = 0 \right\}$ is a subspace of $\mathbf{C}^3$ by applying the three-part test of Theorem TSS. Write your proof according to the standards of this semester's writing exercises.

R.2. [15 points] Suppose that $Z : V \longrightarrow V$ is the linear transformation denoted by $Z(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$ (i.e. Z is the "zero" linear transformation). Suppose that $T : V \longrightarrow V$ is a linear transformation such that $T^4 = Z$ (where $T^4 = T \circ T \circ T \circ T$). Use a proof by contradiction to prove that T is not invertible. Write your proof according to the standards of this semester's writing exercises.

Staging Area

Computations

Not Seen before

1. Find a basis for the kernel of the linear transformation $T : M_{2,2} \to M_{2,2}$ defined by $T(A) = \frac{1}{2}A - \frac{1}{2}A^t$.
2. ***
3. If T is conjugate to the identity map then T is an isomorphism.
4. It is "obvious" that if $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0}$ is a nontrivial relation of linear dependence and if $a_i \neq 0$, then $\vec{v}_i$ is in the span of the remaining vectors. Use this fact to
5. *** Define $\vec{v} + \ker(T)$ and have students show the set of all such is a vector isomorphic to $T(V)$.

(a) Show $\vec{v}_1 + \ker(T) = \vec{v}_2 + \ker(T)$ if and only if $\vec{v}_2 - \vec{v}_1 \in \ker(T)$

(b) Show well-defined

(c) Show injective

(d) Show surjective

6. *** Show that T is surjective iff $\ker(z) \subset R(T)$

C.4. Let $S : P_2 \longrightarrow P_3$ be given by $S(p) = x^3 p'' - x^2 p' + 3p$. Find the matrix representation of S with respect to the bases B, C where the basis for P_2 is $B = \left\{x+1, x+2, x^2\right\}$ and the basis for P_3 is $C = \left\{1, x, x^2, x^3\right\}$.