Mathematics 290 **Exam 5 and Final Examination** **Fall 2006**

December 15, 2006

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Name

Technology used: _______________

Directions:

Directions:

- **Only write on one side of each page.**
- **Use terminology correctly.**
- **Partial credit is awarded for correct approaches so justify your steps.**

Exam 5

"Computational" Problems

C.1. Do **one** (1) of the following:

(a) Given the matrix $A = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ where p and q are real numbers with $q \neq 0$.

i. Show that the eigenvalues of A are $\lambda_1 = p + iq$ and $\lambda_2 = p - iq$.

ii. Determine a nonsingular matrix S and a diagonal matrix D for which $S^{-1}AS = D$.

(b) Prove that $T : P_2 \to P_2$ defined by $(Tf)(x) = f(x+1)$ is both a linear transformation and injective.

C.2. Find the matrix $M^T_{B,B}$ of the linear transformation $T : P_2 \to P_2$ defined by $(Tf)(x) = f(x+1)$ where $B = \{1, x, x^2\}$ is the standard basis of P_2.

Do Two (2) of these "In text, class or homework" problems

M.1. Prove the third part (transitive property) of Theorem SER, Similarity is an Equivalence Relation:

Suppose A, B and C are square matrices of size n. Then

(a) A is similar to A. (Reflexive)

(b) If A is similar to B, then B is similar to A. (Symmetric)

(c) If A is similar to B and B is similar to C, then A is similar to C. (Transitive)

M.2. Prove Theorem EDELI, Eigenvectors with Distinct Eigenvalues are Linearly Independent:

Suppose that A is a square matrix and $S = \{x_1, x_2, x_3, ..., x_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then S is a linearly independent set.

M.3. Prove Theorem SSRLT, Spanning Set for Range of a Linear Transformation

Suppose that $T : U \to V$ is a linear transformation and $S = \{u_1, u_2, u_3, ..., u_t\}$ spans U. Then $R = \{T(u_1), T(u_2), T(u_3), ..., T(u_t)\}$ spans $R(T)$.

M.4. Prove Theorem VRI, Vector Representation is Injective

If $B = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is a basis for the vector space V then The function $\rho_B : V \to \mathbf{C}^n$ given in Definition VR [548] is an injective linear transformation.

Do One (1) of these "Not in Text" problems

T.1. Prove: If A is diagonalizable, then A^T is similar to A.

T.2. This problem is Theorem CLTLT, Composition of Linear Transformations is a Linear Transformation in the textbook. Prove it, using the definition of linear transformation (you cannot just cite a theorem in the book.)

T.3. Define two vectors f, g in the vector space P_2 to be **orthogonal with respect to the coordinate basis** $B = \left\{1, x, x^2\right\}$ if $\langle \rho_B(f), \rho_B(g) \rangle = 0$. [Recall that $\rho_B(f)$ is a vector in $\mathbf{C}^3$.] Find a basis for the set of all polynomials g in P_2 that are orthogonal with respect to the coordinate basis B to the polynomial $f(x) = 2 + x$.

Cumulative Exam

Do Two (2) of these "In text, class or homework" problems

CC.1. (1 point each) If A is a square matrix, make a list of statements equivalent to "A is nonsingular"

CC.2. Let U, V be abstract vector spaces and $T : U \to V$ a function. Show that T is a linear transformation **if and only if** for all $\vec{u}_1, \vec{u}_2 \in U$ and all scalars a, b we have $T(a\vec{u}_1 + b\vec{u}_2) = aT(\vec{u}_1) + bT(\vec{u}_2)$.[Be sure to prove **both** directions of the "if and only if".]

CC.3. Given an invertible matrix S, prove the following transformation $T : M_{nn} \to M_{nn}$ is linear.

$$T(A) = S^{-1}AS$$

CC.4. If there are square matrices A and B satisfying the property that $B^2 = A$, then we say B is a **square root** of A. It is easy to see that a diagonal matrix $D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d_{nn} \end{bmatrix}$ has $\begin{bmatrix} \sqrt{d_{11}} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sqrt{d_{nn}} \end{bmatrix}$ as a square root.

Prove that if A is a diagonalizable matrix, then A has a square root.

Do One (1) of these "Not in text" problems

MM.1. It is "obvious" that if $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0}$ is a nontrivial relation of linear dependence and if $a_i \neq 0$, then $\vec{v}_i$ is in the span of the remaining vectors. Use this fact to prove that if a set $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \cdots, \vec{v}_n\}$ is linearly dependent, then there is an index t for which $\vec{v}_t$ is equal to a linear combination of the vectors $\vec{v}_{t+1}, \vec{v}_{t+2}, \cdots, \vec{v}_n$ that **follow** it in S.

MM.2. Use the principle of mathematical induction to prove the following fact we have used repeatedly throughout the semester.

Suppose V is a vector space, $\vec{v}_1, \vec{v}_2, \vec{v}_3, \cdots, \vec{v}_n$ and $\vec{u}_1, \vec{u}_2, \vec{u}_3, \cdots, \vec{u}_n$ are vectors in V. Then $(\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n) + (\vec{u}_1 + \vec{u}_2 + \cdots + \vec{u}_n) = (\vec{v}_1 + \vec{u}_1) + (\vec{v}_2 + \vec{u}_2) + \cdots + (\vec{v}_n + \vec{u}_n)$ for every positive integer n.

You MUST do both of these problems.

Show your work on this page.

1. (10 points) Prove that the set $Z = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \, 2x_1 - 4x_2 + x_3 = 0 \right\}$ is a subspace of $\mathbf{C}^3$ by applying the three-part test of Theorem TSS.

2. (10 points) Suppose that A and B are square matrices of the same size, and AB is nonsingular. Give a proof by contradiction that B is nonsingular. (Do not do this problem simply by quoting a theorem from the book.)