Exam 4

I affirm this work abides by the university's Academic Honesty Policy.

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Print Name, then Sign

Directions:

- **Only write on one side of each page.**
- **Use terminology correctly.**
- **Partial credit is awarded for correct approaches so justify your steps.**

Do all three (3) of these "Computational" problems

C.1. [15 points] Given the matrix $A = \begin{bmatrix} 25 & -8 & 30 \\ 24 & -7 & 30 \\ -12 & 4 & -14 \end{bmatrix}$. Find an invertible matrix S with integer entries, the inverse matrix S^{-1} and a diagonal matrix D for which $S^{-1}AS = D$.

C.2. [15 points] Show that the function $T : P_4 \longrightarrow P_2$ defined by $T(p) = p''(x)$ is a linear transformation.

C.3. [5, 5 points] Define a linear transformation $T : \mathbf{C}^2 \to \mathbf{C}^2$ by $T\left(a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = a \begin{bmatrix} 9 \\ 3 \end{bmatrix} + b \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

1. Find the matrix representation $M^T_{B,C}$ where $B = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ and $C = \left\{ \begin{bmatrix} 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$.
2. Explain why T is invertible.

Do any two (2) of these "In Class, Text, or Homework" problems

M.1. [15 points] Prove If A is diagonalizable then A^3 is diagonalizable.

M.2. [15 points] Suppose $T : U \to V$ and $S : U \to V$ are linear transformations and recall that the function $T + S : U \to V$ is defined by $(T + S)(\vec{u}) = T(\vec{u}) + S(\vec{u})$ for all $\vec{u} \in U$. Prove that $T + S$ is a linear transformation.

M.3. [15 points] Prove that if W is a subspace of V and $T : U \to V$ is a linear transformation, then $T^{-1}(W) = \{\vec{u} \in U : T(\vec{u}) \in W\}$ is a subspace of U.

Do any two (2) of these "Other" problems

T.1. [15 points] Prove that if X is a subspace of U and $T : U \to V$ is a linear transformation, then the set $T(X) = \{T(\vec{x}) \in V : \vec{x} \in X\}$ is a subspace of V.

T.2. [10, 5 points] Suppose U, V, and W are vector spaces and $T : U \longrightarrow V$, $S : V \longrightarrow W$ are linear transformations.

1. Prove $\ker(T) \subseteq \ker(S \circ T)$.
2. If B,C, D are bases of U, V, W respectively, what result about the matrices $M^T_{B,C}$, $M^S_{C,D}$ and $M^{S\circ T}_{B,D}$ corresponds to $\ker(T) \subseteq \ker(S \circ T)$?

T.3. [15 points] Suppose $T : U \to V$ and $S : V \to U$ are linear transformations and $\dim(U) > \dim(V)$ .Prove $S \circ T$ cannot be the identity map $I_U : U \to U$. [Hint: the identiy map is injective.]

T.4. [15 points] Suppose U and V are vector spaces where $B = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is a basis for U and $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n, \mathbf{v}_{n+1}, \cdots, \mathbf{v}_m\}$ is a basis for V where $n < m$. Define linear transformations $T : U \to V$ and $S : V \to U$ where $S \circ T = I_U$ (Recall that $I_U : U \to U$ is the identity map).