Exam 4

I affirm this work abides by the university's Academic Honesty Policy.

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Print Name, then Sign

Directions:

- **Only write on one side of each page.**
- **Use terminology correctly.**
- **Partial credit is awarded for correct approaches so justify your steps.**

You must do this problem.

D.1. [1 point each] Given a set V and an addition and scalar multiplication for elements in V, there are 10 properties that must hold for V to be a vector space. List those properties. Give mathematical statements of the properties not the names of the properties.

Do two (2) of these "Computational" problems

C.1. [15 points] Show that the set of all vectors in $\mathbf{C}^4$ whose coordinates sum to zero is a subspace of $\mathbf{C}^4$.

C.2. [15 points] Prove that the set $S = \left\{1 + t^2, t + t^2, 1 + 2t - t^2\right\}$ is a basis for P_2.

C.3. [15 points] Let V be the subspace of $\mathbf{C}^4$ with basis $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$. Find a basis for $V^{\perp}$.

Do any two (2) of these "In Class, Text, or Homework" problems

M.1. [8, 7 points each] Do both of the following.

1. Give an example of a 2×2 matrix A for which $A \neq O_2$ and $A^2 = O_2$.
2. Given that A and B are both $n \times n$ matrices. Prove that if $\vec{x}$ is an eigenvector for both A and B, then $\vec{x}$ is an eigenvector for $A + B$

M.2. [15 points] Given a basis $\{\vec{v}_1, \cdots, \vec{v}_n\}$ of $\mathbf{C}^n$ and a matrix A for which $\{\vec{v}_{r+1}, \cdots, \vec{v}_n\}$ is a basis for the null space of A. Prove the set $\{A\vec{v}_1, \cdots, A\vec{v}_r\}$ spans the column space of A.

M.3. [15 points] Let U be a vector space and V, W subspaces of U of dimension 2 and 3 respectively. Let $B = \{\vec{v}_1, \vec{v}_2\}$ be a basis of V and $C = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ a basis for W.If the only vector common to both V and W is $\vec{0}$ prove that the set $B \cup C = \{\vec{v}_1, \vec{v}_2, \vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is linearly independent.

Do any two (2) of these "Other" problems

T.1. [12, 3 points each] A square matrix of size n is said to be **nilpotent** if there is a positive integer s for which $A^s = O_n$. Prove that if A is nilpotent, then

1. $\lambda = 0$ is an eigenvalue of A and is the only eigenvalue of A.
2. Use this information to determine the characteristic polynomial of A, $p_A(x)$.

T.2. [15 points] Prove that if U and W are both subspaces of a vector space V then the intersection, $U \cap W$, is also a subspace of V.

T.3. [3 points each] Suppose V is a vector space of dimension 7 and W is a subspace of dimension 4.

1. (a) **True or False**

   (b) Every basis for W can be extended to a basis for V by adding three more vectors.

   (c) Every basis for V can be reduced to a basis for W by removing three vectors.
   Suppose now that $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5, \vec{v}_6, \vec{v}_7, \vec{v}_8, \vec{v}_9$ are nine vectors in $\mathbf{C}^7$.
   Choose your answer.

   (d) Those vectors (are)(are not)(might be) linearly independent.

   (e) They (do)(do not)(might) span $\mathbf{C}^7$.

   (f) If those vectors are the columns of matrix A, then $A\vec{x} = \vec{b}$ (has)(does not have)(might not have) a solution.