Mathematics 181 – **B** **December** 17, 2007 **Exam** 5/**Final**

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Name

Technology used: ___________________________

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use a text or notes or technology. Include a careful sketch of any graph obtained by technology in solving a problem. **Only write on one side of each page.**

Examination

Do any six (6) of the following problems (15 points each)

1. (3 points each)
   (a) Give an example of a convergent infinite series.
   (b) Give an example of a divergent infinite sequence.
   (c) Give an example of a monotone, non-increasing, unbounded sequence.
   (d) Give an example of a converging improper integral.
   (e) The first few terms of the Taylor series at $x = -3$ of a function $f(x)$are given by $f(x) = 3 + 4(x+3)^2 + 9(x+3)^3 - 20(x+3)^5 + (x+3)^8 + \cdots$. What is the value of $f^{(5)}(-3)$?

2. Choose **one** (1) of the following infinite series and determine whether or not it converges. Specify which test(s) you use and show the work justifying your conclusion.
   (a) $\sum_{n=2}^{\infty} \frac{n}{\left(\sqrt{\ln(n)}\right)^n}$
   (b) $\sum_{n=0}^{\infty} \frac{(3n)!}{n!(n+1)!(n+2)!}$

3. Does the infinite series $\sum_{n=1}^{\infty} (-1)^{n-1}\left(\sqrt[n]{321}\right)$ converge absolutely, conditionally, or neither? Specify which test(s) you use and show all of your work.

4. The convergent series
$$\frac{1}{5} - \frac{1}{3} + \frac{1}{25} - \frac{1}{9} + \frac{1}{125} - \frac{1}{27} + \cdots + \frac{1}{5^n} - \frac{1}{3^n} + \cdots$$
does not meet one of the conditions of the Alternating Series Test. Determine which one it fails and then find the exact sum of the series.

5. Given the power series $\sum_{n=0}^{\infty} \frac{\left(x-\sqrt{2}\right)^{2n+1}}{2^n}$,
   (a) Find the radius of convergence and the interval of convergence.
   (b) For which numbers x does the series converge absolutely? For which numbers does it converge conditionally?

6. Let $P_n(x)$ be the Taylor polynomial of order n for the function $f(x) = e^{\frac{1}{2}x}$. Find the least integer n for which $P_n(4)$ approximates e^2 to within 0.01. [Use the Useful Information section below and the fact that $e^2 < 10$.]

7. Use substitution into the Taylor series of known functions [See Useful Information below] to find the first five non-zero terms of the Taylor series at $x = 0$ of $f(x) = x\cosh(x)$. Be sure to simplify the terms and to specify on which interval you know this series converges absolutely.

8. The formula for computing a Taylor series for a function f at $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

Use this formula to compute the Taylor series with $a = 1$ for the function $f(x) = \ln(x)$.

Cumulative Final Examination

Section One (15 points each): Do any two (2) of the following.

I.1. Use Riemann sums to carefully and fully explain why $\lim_{n\to\infty} \frac{1}{n}\left[\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{3\pi}{n}\right) + \cdots + \sin\left(\frac{n\pi}{n}\right)\right]$ is equal to $\int_0^1 \sin(\pi x)\,dx$ and evaluate the limit .

I.2. Find $F'(x)$ if $F(x) = \int_{x^2}^{4} \ln\left(t^2+1\right) dt$.

I.3. Give an ε-N argument that

$$\lim_{n\to\infty} \frac{2n+5}{n+3} = 2.$$

Section Two (15 points each): Do any two (2) of the following.

II.1. Find the volume of the solid generated by revolving the region bounded by the parabola $y^2 = 4x$ and the line $y = x$ about the line $x = 4$.

II.2. Find the length of the curve given by $x = (y^3/12) + (1/y)$, $1 \le y \le 2$.

II.3. Californium-252 is a radioactive isotope used to treat brain cancer and detect explosives in luggage. The half-life of Californium-252 is 2.645 years. How long will it take for 2/5 of a sample to disintegrate?

II.4. Without using a table of integrals or a calculator, use integration to show that the area of the portion of the circle $x^2 + y^2 = 4$ that lies in the first quadrant is π.

Section Three (15) points each: Do any two (2) of the following.

III.1. Evaluate the improper integral for $\Gamma(2)$ where the Gamma function is defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ for any number x in the interval $(0, \infty)$.

III.2. $\int \frac{e^{\arcsin(\sqrt{x})}}{\sqrt{x-x^2}}\,dx$

III.3. $\int \frac{4x+4}{x^2(x^2+4)}\,dx$

Section Four (10 points): Do any one (1) of the following.

IV.1. Explain fully why if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of non-negative numbers, then $\sum_{n=1}^{\infty} a_n b_n$ must also converge.

IV.2. Suppose we know that the power series $\sum_{n=1}^{\infty} a_n (x+1)^n$ converges at the value $x = 3$. Explain fully why the series must also converge at $x = -4$.

Useful Information

- $\cosh(x) = \frac{1}{2}\left[e^x + e^{-x}\right].$

Taylor's Formula

If f has derivatives of all orders in an open interval I containing the number a, then for each positive integer n and for each x in I, we have $f(x) = P_n(x) + R_n(x)$ where $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}$ and $R_n(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$ for some number c between a and x.

Frequently Used Taylor Series

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, for $|x| < 1$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, for $|x| < \infty$
- $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, for $|x| < \infty$
- $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, for $|x| < \infty$
- $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$, for $-1 < x \leq 1$