Turn In Problems 2.1

2.1 Exercise 22 on page 324 (Section 5.1)

1. Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of one of the n congruent triangles formed by drawing radii to the vertices of the polygon.

   Each of the n triangles is isosceles with summit angle $\frac{2\pi}{n}$ and unit side lengths. If we drop a perpendicular from the summit to the base we obtain two congruent right triangles whose adjacent side has length $1\cos\left(\frac{\pi}{n}\right)$ and opposite side has length $1\sin\left(\frac{\pi}{n}\right)$. Thus the area of a single isosceles triangle is

$$\begin{aligned}(2)\left(\frac{1}{2}\right)\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right) &= \sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)\\ &= \frac{1}{2}\sin\left(\frac{2\pi}{n}\right)\end{aligned}$$

   where the last equality comes from the trigonometric identity: $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.

2. Since there are n of these triangles, the total area contained in the regular n-sided polygon is

$$\begin{aligned}A_p &= n\left[\frac{1}{2}\sin\left(\frac{2\pi}{n}\right)\right]\\ &= \frac{1}{2}\frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{1}{n}}\end{aligned}$$

   Since this last expression has the form '$\frac{0}{0}$' as n limits to infinity we apply L'Hospitals rule to compute the following limit

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{2}\frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{1}{n}} &= \frac{1}{2}\lim_{n\to\infty}\frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{1}{n}}\\ &= \frac{1}{2}\lim_{n\to\infty}\frac{\cos\left(\frac{2\pi}{n}\right)\cdot\left(\frac{-2\pi}{n^2}\right)}{\left(\frac{-1}{n^2}\right)}\\ &= \frac{1}{2}\lim_{n\to\infty}\left[2\pi\cos\left(\frac{2\pi}{n}\right)\right]\\ &= \left(\frac{1}{2}\right)(2\pi)\lim_{n\to\infty}\cos\left(\frac{2\pi}{n}\right)\\ &= \left(\frac{1}{2}\right)(2\pi)\cos(0)\\ &= \pi\end{aligned}$$

3. If we use a circle of radius r instead of radius 1 in the above computations all that will change in part 1. is that we replace $\sin\left(\frac{\pi}{n}\right)$ and $\cos\left(\frac{\pi}{n}\right)$ with $r\sin\left(\frac{\pi}{n}\right)$ and $r\cos\left(\frac{\pi}{n}\right)$, respectively.

   This will give us the area of a single isosceles triangle as $(2)\left(\frac{1}{2}\right)r\sin\left(\frac{\pi}{n}\right)r\cos\left(\frac{\pi}{n}\right) = \frac{r^2}{2}\sin\left(\frac{2\pi}{n}\right)$ and the limit we wish to compute is

$$\lim_{n\to\infty}\frac{r^2}{2}\frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{1}{n}}$$

   which is precisely r^2 times the limit we computed in part 2. Thus the final answer in this case is πr^2 as we expected.

2.2 Use the Principle of Mathematical Induction to prove the constant multiple rule

$$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k \qquad \text{(Display 1.)}$$

where c is a constant.

1. First we note that the formula in Display 1 is true when $n = 1$ because

$$\sum_{k=1}^{1} ca_k = (ca_1) = c\,(a_1) = c\sum_{k=1}^{1} a_k$$

2. Now we show that **if** we know that the formula in Display 1 is true when the n is at the top of the Sigma, **then** the formula will also be true when $n+1$ is at the top of the Sigma. Specifically, we show that $\sum_{k=1}^{n+1} ca_k = c\sum_{k=1}^{n+1} a_k$ is a logical consequence of $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$.

   Note that we can break up the sum of the $n+1$ terms in $\sum_{k=1}^{n+1} ca_k$ into the sum of the first n terms plus the very last (the $n+1$ st) term.

$$\sum_{k=1}^{n+1} ca_k = \sum_{k=1}^{n} ca_k + ca_{n+1}$$

   Continuing we have

$$\begin{aligned}
\sum_{k=1}^{n+1} ca_k &= \left(\sum_{k=1}^{n} ca_k\right) + ca_{n+1} \\
&= \left(c\sum_{k=1}^{n} a_k\right) + ca_{n+1} \quad \text{because Display 1 is assumed to hold with } n \text{ at the top} \\
&= c\left(\sum_{k=1}^{n} a_k + a_{n+1}\right) \quad \text{by factoring } c \text{ out of the two terms} \\
&= c\sum_{k=1}^{n+1} a_k \quad \text{since the inside of the parentheses is the sum of all } n+1 \text{ terms.}
\end{aligned}$$

   Thus, we have shown that if the formula in Display 1 holds for some positive integer n, then it also holds for the next positive integer $n+1$. Since we know it holds for the integer 1 we can conclude it holds for every positive integer.