## The Basics of Mathematical Logic

The basic logical objects we will be using in this class are mathematical "logical propositions" — that is, statements that have the property of being either true or false but not both. So, $5 + 2 = 96$ is a logical proposition. If we introduce variables as in $x > 2$, then we don't have a logical proposition because we don't know enough about "x" to decide if the statement is true or false. To overcome this we use quantifiers from Predicate Logic (not Propositional Logic). For example, both of the following are valid logical propositions. The first one is false and the second is true.

$$\begin{aligned} \text{Every real number } x \text{ satisfies } x &> 2. \\ \text{There is a real number } x \text{ satisfying } x &> 2. \end{aligned}$$

Note that saying "every" ( or "for all" or "all") can make a huge difference in the meaning of a logical statement. The same is true for "there is" (or "there are" or "some"). The former are instances of using the **universal quantifier**, written $\forall$ of predicate logic, and the latter are instances of using the **existential quantifier**, written $\exists$ . It is extremely important that you note the occurrence of these quantifiers when you are reading mathematics. Notational shorthand for these quantifiers

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**Logical Relations** There are a few basic rules for manipulating logical statements. These are:

**Negation** Given a logical statement (abbreviated by the letter "p"), then the negation of p is true whenever p itself is false, and the negation of p is false whenever the statement p itself is true. A shorthand notation for the logical proposition "the negation of p" is "$\neg p$".

**Conjunction** Given a logical statement p and another logical statement q, then the single logical statement "p and q" is true only when both p and q are true. Thus, the conjunction "p and q" is false whenever at least one of p, q is false. A shorthand notation for the logical proposition "p and q" is "$p \wedge q$ ".

**Disjunction** Given a logical statement p and another logical statement q, then the single logical statement "p or q" is false only when both p and q are false. Thus, the disjunction "p or q" is true if at least one of p, q is true. For this reason, disjunction is sometimes called "inclusive or" since its truth includes the possibility that both p and q are true. A shorthand notation for the logical proposition "p or q" is "$p \vee q$ ".

**Implication** Given a logical statement p and another logical statement q, then the single logical statement "if p, then q" is false only if p is true and q is false. This is the basis for making logical deductions. Specifically, if you happen to know that the entire implication "if p, then q" is true and if you also happen to know that the logical proposition "p" is true, then implication allows us to deduce that "q" must be true as well. Here is a simple example. The implication is "if Annie can jump over a six foot fence, then Annie can jump over a four foot fence." Note that this is obviously true whether or not Annie can jump over a six foot fence. Now, if Annie happens to be a healthy adult deer, then it is true that she can jump over a six foot fence. We are therefor justified in concluding she can jump over a four foot fence.

A shorthand notation for the implication "if p then q" is "$p \Rightarrow q$ " and different ways of saying it are: "if p, then q", "p implies q", "p's truth is sufficient for q to be true", and "q's truth is necessary for p to be true".

**Equivalence** Given a logical statement p and another logical statement q, then the single logical statement "p if and only if q" is true precisely when p and q are either both true or both false. A shorthand notation for the logical proposition "p if and only if q" is "$p \Leftrightarrow q$ ".

Other ways of saying "p if and only if q" are: (1)"p is equivalant to q" and (2) **both** "p implies q" and "q implies p" are true.

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**Negating Quantifiers** At times the author or I (or you) will wish to negate a mathematical statement so here is an example illustrating that process with quantified statements.

The logical statement "for every real number $x$, $x^2 > 0$" is false so it's negation "it is not the case that for every real number $x$, $x^2 > 0$ " must be true. We can translate the negation into English in two ways:

1. " it is not the case that the square of every real number is greater than zero"
2. "there is a real number, zero, whose square is not greater than zero"

Using $\mathbb{R}$ for the set of real numbers, shorthand notation for the two statements above are, respectively $\neg(\forall x \in \mathbb{R}, \quad x^2 > 0$ and $\exists\ 0 \in \mathbb{R} \quad \neg(0^2 > 0)$.

This illustrates the first of the following logical rules (where we use $p_x$ to denote an arbitrary statement involving the variable $x$):

1. $\neg(\forall x\ p_x) \Leftrightarrow \exists\ x\ \neg p_x$
2. $\neg(\exists x\ p_x) \Leftrightarrow \forall\ x\ \neg p_x$

The first says "it is not the case that $p_x$ is true for all $x$ is equivalent to saying there is an $x$ for which $p_x$ is not true" and the second says "it is not the case that there is an $x$ making $p_x$ true is equivalent to saying that every $x$ makes $p_x$ false".

We can organize our definitions of negation, conjunction, disjunction, implication and equivalence by using "Truth Tables". As one example, here is the truth table for the conjunction $p \wedge q$.

| p | q | p ∧ q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

## Problems on the Basics of Mathematical Logic

1. Explain in your own words why the following (relatively complex) logical statement "(p implies q) is true if and only if (the negation of q implies the negation of p) is true" must **always** be true.

   That is, show the following is **always** true

   $$(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p).$$

2. Explain in your own words why $\neg(\neg p) \Leftrightarrow p$. is **always** true.

3. Explain in your own words why $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$ is **always** true.

4. Explain in your own words why $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$ is **always** true.

5. What can you say about the logical statement $r \wedge \neg r$ ?

6. What is the truth table for $p \Rightarrow q$?

7. What is the truth table for $p \Leftrightarrow q$?

8. Come to class ready to discuss why the following is **always** true.

   $$(p \Rightarrow q) \Leftrightarrow ([(p \wedge \neg q) \Rightarrow (r \wedge \neg r)] \Rightarrow q).$$