Equations of Planes

The geometry of how planes sit in three-dimensional space is very similar to the geometry of how lines sit in two-dimensional space. The following is a quick introduction to the details.

Recall that an equation of a line in the plane is a *linear equation* in two variables. If we use x and y as the variables then we also refer to the plane as the xy-plane. As an example, consider

$$3x + 4y - 8 = 0.$$

This is called a *standard form* for the linear equation. You are also familiar with other forms that result from algebraically modifying the standard form. For example, a *slope-intercept form* is

$$y = -\frac{3}{4}x + 2$$

and the *point-slope form* that uses the point $(-4, 5)$ is

$$y - 5 = -\frac{3}{4}(x + 4).$$

[Note that point-slope forms are quite useful when we are intested in a particular point – say, when we are computing the equation of the tangent line to the graph of the function $y = x^2$ at the point $(-3, 9)$.]

In general, we have

$Ax + By + C = 0$	standard form
$y = mx + b$	slope-intercept form
$y - y_0 = m(x - x_0)$	point-slope form

We use these last two equations to read off geometric information about how the line sits in the plane. Later we will see that the constants A and B in the standard form also have geometric interpretations.

Planes in space are described by *linear equations* in three variables. For example, consider the equation

$$3x + 4y - 2z - 12 = 0.$$

The set of all points with cartesian coordinates (x, y, z) that satisfy this equation form a specific plane. By solving for one of the variables, say z, we can obtain geometric information about this plane. Solving, we have

$$z = \frac{3}{2}x + 2y - 6.$$

This equation is a *slopes-intercept form* for the plane. Note that theire are two slopes: the coefficient $\frac{3}{2}$ is the x-slope and the coefficient 2 is the y-slope for this plane. We denote the x-slope and y-slope by m_x and m_y, respectively and so, for this plane

$$m_x = \frac{3}{2} \quad \text{and} \quad m_y = 2.$$

The geometric interpretation of the x-slope is that it is the "rise over run" of the plane if we hold the y variable constant and, similarly, the y-slope is the "rise over run" if hold x constant. More specificially, we have

$$m_x = \frac{\text{rise in } z}{\text{rise in } x} \quad \text{with } y \text{ held constant}$$

and

$$m_y = \frac{\text{rise in } z}{\text{rise in } y} \quad \text{with } x \text{ held constant.}$$

Geometrically, the two slopes $m_x = \frac{3}{2}$ and $m_y = 2$ tell us how the plane is oriented in space because we know z changes by 3 units whenever x changes by 2 but the y values do not change at all and similarly, z changes by 2 units whenever y changes by 1 unit but the x values do not change. Note also, that when x and y are both 0 then the equation tells us $z = -6$ so the constant term in the slopes-intercept equation is the z-intercept of the plane. Thus the z-intercept picks out a particular plane frome the stack of parallel planes with $m_x = \frac{3}{2}$ and $m_y = 2$.

Similar to the equations of lines in the plane, we can express the equation of a plane in several different ways:

$Ax + By + Cz + D = 0$	standard form
$z = m_x x + m_y y + b$	slopes-intercept form
$z - z_0 = m_x (x - x_0) + m_y (y - y_0)$	point-slopes form

EXAMPLE:

Find the standard form of the equation for the plane that contains the points $P(2, 5, 0)$, $Q(4, 5, 6)$, and $R(2, 3, 4)$.

Since y is constant between P and Q then we deduce that

$$m_x = \frac{6 - 0}{4 - 2} = 3.$$

Similarly, since x is constant between P and R we deduce

$$m_y = \frac{0 - 4}{5 - 3} = -2.$$

Using the point-slopes form with the point $R(2, 3, 4)$ (note that we could have used either P or Q here) we obtain

$$z - 4 = 3(x - 2) - 2(y - 3).$$

Simplifying, we see the standard form is

$$3x - 2y - z + 4 = 0.$$

We can now easily check that all three points satisfy this equation.

Problems on Equations of Planes

1. Determine which, if any of the following points are on the plane having equation $2x-y+6z=14$.

   $P(5,-4,0)$, $Q(1,6,2)$, and $R(2,8,3)$

2. Determine the x-intercept, the y-intercept, and the z-intercept of the plane having equation $2x-y+6z=14$.

3. Determine the slopes m_x and m_y of the plane having equation $2x-y+6z=14$.

4. Find the standard form equation for the plane with slopes $m_x=3$, $m_y=-2$ and containing the point $P(2,-6,1)$.

5. Find an equation for the plane that contains the points $(0,0,0)$, $(2,0,6)$, and $(0,5,20)$.

6. Find an equation for the plane that contains the points $(0,0,0)$, $(0,4,-8)$, and $(3,0,6)$.

7. Find an equation for the plane that contains the points $(1,3,2)$, $(1,7,10)$, and $(3,3,8)$.

8. Find an equation for the plane that contains the points $(7,2,1)$, $(5,2,-4)$, and $(5,-2,10)$.

9. (*Challenge Problem*) Find an equation for the plane that contains the points $(1,3,2)$, $(1,7,10)$, and $(4,2,1)$.

10. (*Challenge Problem*) Find an equation for the plane that contains the points $(1,3,2)$, $(5,7,10)$, and $(4,2,1)$.

Solutions

1. $(5,-4,0), (2,8,3)$ on the plane, $(1,6,2)$ not

2. $(7,0,0), (0,-14,0)$, and $(0,0,7/3)$.

3. $m_x=-1/3$, $m_y=1/6$

4. $3x-2y-z=17$

5. $z=3x+4y$

6. $z=2x-2y$

7. $z=3x+2y-7$ or $z-2=3(x-1)+2(y-3)$ or $\cdots$

8. $z=\frac{5}{2}x-\frac{7}{2}y-\frac{19}{2}$ or $z-1=\frac{5}{2}(x-7)-\frac{7}{2}(y-2)$ or $\cdots$

9. $z=\frac{1}{3}x+2y-\frac{13}{3}$ or $z-2=\frac{1}{3}(x-1)+2(y-3)$ or $\cdots$

10. $z=\frac{1}{4}x+\frac{7}{4}y-\frac{7}{2}$ or $z-2=\frac{1}{4}(x-1)+\frac{7}{4}(y-3)$ or $\cdots$