"*To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty of nature. If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in*". -Richard Feynman (1918-1988)

Multiplication of Partitioned Matrices

It is sometimes useful to break a large matrix down into smaller submatrices by slicing it up with horizontal or vertical lines that go all the way through the matrix. For example, we can think of the 4×4 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 \end{bmatrix}$$

as a 2×2 **partitioned matrix** whose "entries" are four 2×2 matrices:

$$A = \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

with $A_{11} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A_{21} = \begin{bmatrix} 9 & 8 \\ 5 & 4 \end{bmatrix}$, etc. There is no need for the submatrices to be square or of the same size. For example

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \left[\begin{array}{c|c|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array}\right] = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix}.$$

One of the most useful properties of partitioned matrices is how they behave with respect to matrix multiplication. The following theorem describing how to multiply partitioned matrices is given without proof.

Theorem (Partitioned Matrices, Block Multiplication - PMBM**):** Suppose we are given matrices A and B for which the product AB makes sense and which are partitioned into smaller matrices A_{ij} and B_{rs} as shown below. Provided all the matrix products $A_{ik}B_{kj}$ are defined then

$$AB = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rn} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1s} & \cdots & B_{1p} \\ B_{21} & B_{22} & \cdots & B_{2s} & \cdots & B_{2p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{ns} & \cdots & B_{np} \end{bmatrix}$$

is the partitioned matrix whose rs th "entry" is the matrix

$$\begin{aligned} [AB]_{rs} &= A_{r1}B_{is} + A_{r2}B_{2s} + \cdots + A_{rn}B_{ns} \\ &= \sum_{k=1}^{n} A_{rk}B_{ks}. \end{aligned}$$

Homework Problem

Let A be the partitioned matrix

$$A = \left[\begin{array}{cc} A_{11} & A_{12} \\ O_{m\times n} & A_{22} \end{array} \right]$$

where A_{11} is an invertible, nonsingular $n \times n$ matrix, A_{22} is an invertible, nonsingular $m \times m$ matrix, A_{12} is an $n \times m$ matrix, and $O_{m\times n}$ is the zero matrix. Use multiplication of partitioned matrices to show that A is invertible. Specifically, determine the partitioned $(n+m) \times (n+m)$ matrix

$$B = \left[\begin{array}{cc} B_{11} & B_{12} \\ B_{21} & B_{22} \end{array} \right]$$

that satisfies

$$BA = I_{n+m}$$

and then check that $AB = I_{n+m}$ is also satisfied.