Mathematics 180A Exam 3 October 29, 2009

Technology used:______________________________ ____________________
Name

Directions:

- **Only write on one side of each page.**
- **Use terminology correctly.**
- **Show your work: No Work – No Credit**
- **Partial credit is awarded for correct approaches so justify your steps.**

Do any five (5) of the following.

A. [8 points each] Evaluate the following using derivative rules and formulas.

(a) $F'(x)$, if $F(x) = x^4 + [x^3 + (x^2+1)]^3$

$$\begin{aligned} F'(x) &= 4x^3 + 3\left[x^3 + \left(x^2+1\right)\right]^2 \frac{d}{dx}\left[x^3 + \left(x^2+1\right)\right] \\ &= 4x^3 + 3\left[x^3 + \left(x^2+1\right)\right]^2 \left(3x^2 + 2x + 0\right) \end{aligned}$$

(b) y', if $y = \ln(\arctan(x))$,

$$\begin{aligned} y' &= \frac{1}{\arctan(x)} \frac{d}{dx}[\arctan(x)] \\ &= \frac{1}{\arctan(x)} \cdot \frac{1}{x^2+1} \end{aligned}$$

(c) $\frac{d}{dx}[\arctan(x) + \frac{d}{dx}(e^{5x})]$

$$\begin{aligned} \frac{d}{dx}[\arctan(x) + \frac{d}{dx}\left(e^{5x}\right)] &= \frac{d}{dx}[\arctan(x) + 5e^{5x}] \\ &= \frac{1}{x^2+1} + 25e^{5x} \end{aligned}$$

(d) $\frac{dy}{dx}$, if $y = \frac{x^2\left(x^2+1\right)^3(x+1)^5}{(x^2+x)^{11}}$ [Use logarithmic differentiation.]

$$\begin{aligned} \ln y &= 2\ln(x) + 3\ln\left(x^2+1\right) + 5\ln(x+1) - 11\ln\left(x^2+x\right) \\ \frac{1}{y}\frac{dy}{dx} &= \frac{2}{x} + \frac{3}{x^2+1}(2x+0) + \frac{5}{x+1}(1+0) - \frac{11}{x^2+x}(2x+1) \\ \frac{dy}{dx} &= \left(\frac{2}{x} + \frac{6x}{x^2+1} + \frac{5}{x+1} - \frac{11(2x+1)}{x^2+x}\right)\frac{x^2\left(x^2+1\right)^3(x+1)^5}{(x^2+x)^{11}} \end{aligned}$$

(e) $\frac{dy}{dx}$, if $\sin(3x-2y) + x^2y^2 = 3x$ [Use implicit differentiation.]

$$\begin{aligned} \sin(3x-2y) + x^2y^2 &= 3x \\ \cos(3x-2y)\left(3 - 2\frac{dy}{dx}\right) + 2xy^2 + 2x^2y\frac{dy}{dx} &= 3 \end{aligned}$$

$$
\begin{aligned}
3\cos(3x-2y) - 2\cos(3x-2y)\frac{dy}{dx} + 2xy^2 + 2x^2y\frac{dy}{dx} &= 3 \\
2x^2y\frac{dy}{dx} - 2\cos(3x-2y)\frac{dy}{dx} &= 3 - 2xy^2 - 3\cos(3x-2y) \\
\left(2x^2y - 2\cos(3x-2y)\right)\frac{dy}{dx} &= 3 - 2xy^2 - 3\cos(3x-2y) \\
\frac{dy}{dx} &= \frac{3 - 2xy^2 - 3\cos(3x-2y)}{2x^2y - 2\cos(3x-2y)}
\end{aligned}
$$

(f) $\left.\frac{dy}{dt}\right|_{t=2}$, if $y = 2u^2 + u$, $u = \tan(x)$, and $x = 3t+1$

$$
\begin{aligned}
\frac{dy}{du} &= 4u+1, \quad \frac{du}{dx} = \sec^2(x), \quad \frac{dx}{dt} = 3 \\
\frac{dy}{dt} &= \frac{dy}{du}\frac{du}{dx}\frac{dx}{dt} \\
&= (4u+1)\cdot\sec^2(x)\cdot 3 \\
&= (4\tan(3t+1)+1)\cdot\sec^2(3t+1)\cdot 3
\end{aligned}
$$

(g) $\frac{dy}{dx}$, if $y = \sinh^{-1}(2\sqrt{x})$

$$
\begin{aligned}
y &= \sinh^{-1}(2\sqrt{x}) \\
\frac{dy}{dx} &= \frac{1}{\sqrt{1+(2\sqrt{x})^2}}\frac{d}{dx}[2\sqrt{x}] \\
&= \frac{1}{\sqrt{1+4x}}\frac{d}{dx}\left[2x^{1/2}\right] \\
&= \frac{1}{\sqrt{1+4x}}\cdot x^{-1/2} \\
&= \frac{1}{\sqrt{x}\sqrt{1+4x}}
\end{aligned}
$$

Do any four (4) of the following

1. [15 points] What is the linearization, $L(x)$, of $f(x) = \sinh(x) + \cosh(x)$ at $x = 0$?

   (a) Solution: $\frac{d}{dx}[\sinh(x)] = \cosh(x) = \frac{1}{2}(e^x + e^{-x})$ and $\frac{d}{dx}[\cosh(x)] = \sinh(x) = \frac{1}{2}(e^x - e^{-x})$ so $f(0) = \sinh(0) + \cosh(0) = 0 + 1$ and $f'(0) = \cosh(0) + \sinh(0) = 1$.This gives

$$
\begin{aligned}
L(x) &= f(0) + f'(0)(x-0) \\
&= 1 - (x-0) \\
&= 1 - x
\end{aligned}
$$

2. [15 points] A television camera is positioned 4000 feet from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Let's assume the rocket rises vertically and its speed is 600 ft/s when it has risen 3000 feet. If the television camera is always aimed at the rocket, how fast is the camera's angle of elevation changing at that same moment?

   (a) **Possible Problem on Next In-Class Quiz. [Hint: use a trigonometric function to relate the height and the angle.]**

3. [15 points] Water drains from the conical tank shown in the the figure on the board at the rate of 4 ft^3/min. How fast is the water level dropping when $h = 6$ ft? [The top of the cone is a circle of radius 4 ft and the total height of the cone is 10 ft. The volume of a cone is $\frac{1}{3}\pi\,(\text{height})\,(\text{radius})^2$.]

   (a) **Possible Problem on Next In-Class Quiz. [Hint: Use similar triangles to relate the height and radius of the water volume.]**

4. [15 points] How accurately should you measure the edge of a cube of side length x to be reasonably sure of calculating the cube's surface area with an error of no more than 2%?

   (a) Solution: If we set $y = 6x^2$ then the error in the surface area is approximately $dy \approx f'(x)\,dx = 12x\;dx$ where dx represents the error in measuring the side. We are given that the relative error in measuring the area is $\frac{dy}{y} < 0.02$ and want to know what the corresponding relative error in the side length $\frac{dx}{x}$ is. So

$$\begin{aligned} \frac{dy}{y} &< 0.02 \\ \frac{12x\;dx}{6x^2} &< 0.02 \\ 2\frac{dx}{x} &< 0.02 \\ \frac{dx}{x} &< 0.01 \end{aligned}$$

   This tells us that we need to measure the side length accurate to within 1% to guarantee surface area is accurate to within 2%.

5. [15 points] Use differentials to approximate $\cos\left(\frac{101\pi}{600}\right)$.[Hint: use $f(x) = \cos(x)$ and $a = \frac{\pi}{6}$.]

   (a) Solution: The approximation formula is $f(a + dx) \approx f(a) + f'(a)\,dx$. We are given $a = \frac{100\pi}{600} = \frac{\pi}{6}$ so we want dx to be $\frac{\pi}{600}$.Then, since $\frac{d}{dx}[\cos(x)] = -\sin(x)$, we have

$$\begin{aligned} f(a + dx) &\approx f(a) + f'(a)\,dx \\ \cos\left(\frac{\pi}{6} + \frac{\pi}{600}\right) &\approx \cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right)\frac{\pi}{600} \\ &= \frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{\pi}{600} \\ &= \frac{\sqrt{3}}{2} - \frac{\pi}{1200} \end{aligned}$$