Mathematics 181A **Fall 2**

Exam 1

September 18, 2008

____
KEY

Technology used: ____ **Only write on one side of each page.**

- Show all of your work. Calculators may be used for numerical calculations and answer checking only.

1. [10 points] Do **one** (1) of the following.

1. A point P in the first quadrant lies on the graph of the function $f(x) = \sqrt[3]{x}$. Express the x-coordinate of P as a function of the slope of the line joining P to the origin.

   Solution: The slope from the point $P(x, \sqrt[3]{x})$ to the origin is $m = \frac{\sqrt[3]{x}-0}{x-0} = \frac{1}{x^{2/3}} = x^{-2/3}$. This means that $x = \left(x^{-2/3}\right)^{-3/2} = m^{-3/2}$.Thus $x = f(m) = m^{-3/2}$.

2. If a composite $f \circ g$ is one-to-one, must g be one-to-one? Explain your answer.

   Solution: Suppose that g is not one-to-one. Then there are numbers x_1, x_2 with $x_1 \neq x_2$ for which $g(x_1) = g(x_2)$.Thus, $(f \circ g)(x_1) = f(g(x_1)) = f(g(x_2)) = (f \circ g)(x_2)$. But this can't happen because we are told that $f \circ g$ is one-to-one. We can therefor deduce that g can't be one-to-one.

2. [15 points] Rewrite the following sum as indicated.

Solution: We make the change of index $j = k + 11$ which tells us that $k = j - 11$ then filling in the missing information we get

$$\begin{aligned} \sum_{k=4}^{101} (2k-1)^2 &= \sum_{j=15}^{101+11} (2(j-11)-1)^2 \\ &= \sum_{j=15}^{112} (2j-23)^2 \end{aligned}$$

3. [15 points] Do **one** (1) of the following. Show your work.

1. Evaluate $\int \frac{1}{t^3}\left(t^2 - 3t^5 + t^{1/2} + 5t^3 \sec^2(t) + 6t^3 \sec(t)\tan(t) + \frac{t^3}{\sqrt{1-t^2}}\right)\, dt$

   Solution: Multiplying through by $\frac{1}{t^3}$ and using stardard antiderivative formulas, we get $\int \left(\frac{1}{t} - 3t^2 + t^{-5/2} + 5\sec^2(t) + 6\sec(t)\tan(t) + \frac{1}{\sqrt{1-t^2}}\right)\, dt = \ln|t| - t^3 + \frac{t^{-3/2}}{-3/2} + 5\tan(t) + 6\sec(t) + \arcsin(t) + C$

2. By differentiating the right hand side, verify the formula $\int \frac{\arctan(x)}{x^2}\, dx = \ln(x) - \frac{1}{2}\ln(1+x^2) - \frac{\arctan(x)}{x} + C$

   Solution: $\frac{d}{dx}\left[\ln(x) - \frac{1}{2}\ln(1+x^2) - \frac{\arctan(x)}{x} + C\right] = \frac{1}{x} - \frac{1}{2}\frac{1}{1+x^2}(2x) - \frac{\frac{1}{1+x^2}(x) - (1)\arctan(x)}{x^2} = \frac{x^2+1}{x(x^2+1)} - \frac{x^2}{x(x^2+1)} - \frac{1}{x(x^2+1)} + \frac{\arctan(x)}{x^2} = 0 + \frac{\arctan(x)}{x^2}$.

4. [8, 7 points] The following is a Riemann sum for a function f with domain an interval $[a, b]$. [Do NOT simplify this sum.]

$$\sum_{k=1}^{n}\left[3\left(5+\frac{6k}{n}\right)^{7}-\left(5+\frac{6k}{n}\right)^{2}+6\right]\frac{6}{n}.$$

1. What is this specific $f(x)$?
2. What is the specific interval $[a, b]$?

   Solution: This Riemann sum has the form $\sum_{k=1}^{n} f(c_k)\,\Delta x$ which tells us that $\Delta x = \frac{6}{n}$ so we know that whatever $[a, b]$ is, we must have $b - a = 6$. We also see that the $\frac{6k}{n}$ terms look like $k\Delta x$ so we deduce that $c_k = 5 + k\Delta x = 5 + \frac{6k}{n}$.this also tells us that our partition starts at $a = 5$ and contains $5 + 1\Delta x$, $5 + 2\Delta x$, etc. Thus we have $f(x) = 3x^7 - x^2 + 6$ with $[a, b] = [5.11]$. An equally valid answer is $f(x) = 3(5+x)^7 - (5+x)^2 + 6$ where the interval is $[0, 6]$.

5. [15 points] Find the derivative of $G(x) = \int_{x^4}^{x} e^{t^2}\,dt$ using part 1 of the Fundamental Theorem of Calculus.

Solution: $G(x) = \int_{x^4}^{0} e^{t^2}dt + \int_{0}^{x} e^{t^2}dt = -\int_{0}^{x^4} e^{t^2}dt + \int_{0}^{x} e^{t^2}dt = -F(x^4) + F(x)$ where $F(x) = \int_0^x e^{t^2}dt$ then the Fundamental Theorem of Calculus, Part 1 tells us that $F'(x) = e^{x^2}$.Now, using the Chain Rule, we have: $G'(x) = -F'(x^4)\,4x^3 + F'(x) = -4x^3e^{x^8} + e^{x^2}$.

6. [15 points each] Do **both** of the following.

1. Evaluate $\int (2t + 1 + 2\cos(2t+1))\ dt = t^2 + 1 + \sin(2t+1) + C.$

   Solution: $\int (2t + 1 + 2\cos(2t+1))\ dt = \int (2t+1)\,dt + \int 2\cos(2t+1)\,dt = t^t + t + \int 2\cos(2t+1)\,dt.$ Using the substitution $u = 2t + 1$, $du = 2\ dt$ on this last integral we get $\int 2\cos(2t+1)\,dt = \int \cos(u)\ du = \sin(u) + C = \sin(2t+1) + C.$

2. Evaluate $\int \frac{(\ln(x+1))^2}{x+1}\ dx = \frac{(\ln(x+1))^2}{3} + C$

   Solution: Using the substitution $u = \ln(x+1)$ we get $du = \frac{1}{x+1}dx$ so that $\int \frac{(\ln(x+1))^2}{x+1}\ dx = \int u^2 du = \frac{1}{3}u^3 + C = \frac{(\ln(x+1))^2}{3} + C.$