Mathematics 433 Fall 2007 Problems to Turn in: 8

Due: November 30, 2007

Name

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. **Only write on one side of each page.**

"*A life spent making mistakes is not only more honorable, but more useful than a life spent doing nothing.*"
– George Bernard Shaw

Problems

1. Do both of the following:

   (a) Prove that O is not a normal subgroup of M.

   (b) Let SM denote the subset of orientation-preserving motions of the plane. Prove SM is a normal subgroup of M and determine its index in M.

2. For those of you who know a bit of complex variables.

   (a) Write the formulas for the motions t_a, ρ_θ and r in terms of the complex variables $z = x + iy$.

   (b) Show every motion has the form $m(z) = \alpha z + \beta$ or $m(z) = \alpha \overline{z} + \beta$, where α, β are complex numbers with $|\alpha| = 1$.

   (c) Find an isomorphism from the group SM to the subgroup of $GL(2, \mathbf{C})$ of matrices of the form $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ with $|a| = 1$.

3. With each of the patterns shown on the sheet of figures labelled "Problem 8.3", find a pattern with the same type of symmetry as those on the accompanying handout (the page numbered 173).

4. Given the subgroup $H = \{1, x^5\}$ of the dihedral group D_{10}.

   (a) Explicitly compute the cosets of H in D_{10}.

   (b) Prove that D_{10}/H is isomorphic to D_5.

   (c) Is D_{10} isomorphic to $D_5 \times H$?

5. List all symmetries of the following figures (found on the last page of the extra-reading handout on Linear Algebra: Orthogonal Matrices and Translations.

   (a) Figure 1.4

   (b) Figure 1.5

   (c) Figure 1.6

   (d) Figure 1.7

6. Prove every finite subgroup of M is a conjugate subgroup of one of the standard subgroups listed in the corollary to the Classification of Finite Symmetry Groups Theorem stated below.

(a) **Corollary 1** *Let G be a finite subgroup of the group of motions M . If coordinates are introducted suitably, then G becomes one of the groups C_n or D_n, where C_n is generated by ρ_θ, $\theta = 2\pi/n$ and D_n is generated by ρ_θ and r.*

7. Find all proper normal subgroups N and identify the corresponding quotient groups D_k/N of the groups D_{13} and D_{15}.

8. Let G be a subgroup of M that contains rotations about two different points. Prove algebraically that G contains a translation.

9. Prove the group of symmetries of the frieze pattern

$$\cdots \mathrm{EEEEEEEE} \cdots$$

is isomorphic to the direct product $C_2 \times C_\infty$ of a cyclic group of order 2 and an infinite cyclic group.

10. Let G be the group of symmetries of the frieze pattern

$$\cdots \subset \supset \subset \supset \subset \supset \cdots$$

(a) Determine the point group $\bar{G}$ of G.

(b) For each element $\bar{g}$ of $\bar{G}$, and each element g of G which represents $\bar{g}$, describe the action of g geometrically.

(c) Let H be the subgroup of translations in G. Determine $[G : H]$.

11. Let G be a discrete group in which every element is orientation-preserving. Prove the point group $\bar{G}$ is a cyclic group of rotations and there is a point p in the plane such that the set of group elements which fix p is isomorphic to $\bar{G}$.

12. Recall that M is the group of rigid motions of the two-dimensional plane. In this problem you investigate the rigid motions of a one-dimensional line.

Let N denote the group of rigid motions of the line $l = \mathbf{R}^1$.Some elements of N are

$$t_a \text{ where } t_a(x) = x + a \text{ and } s \text{ where } s(x) = -x.$$

(a) Show that $\{t_a, t_a s : a \in \mathbf{R}^1\}$ are all of the elements of N,and describe their actions on l geometrically. [Note that $|N|$ is infinite since there is a distinct t_a for each real number a.]

(b) Compute the products $t_a t_b$, $s t_a$, ss.

(c) Find all discrete subgroups of N which contain a translation. It will be convenient to choose your origin and unit length with reference to the particular subgroup. Prove your list is complete.

13. Prove

(a) If the point group of a lattice group G is $\bar{G} = C_6$, then $L = L_G$ is an equilateral triangular lattice, and G is the group of all rotational symmetries of L about the lattice points.

(b) If the point group of a lattice group G is $\bar{G} = D_6$, then $L = L_G$ is an equilateral triangular lattice, and G is the group of all symmetries of L.

Figure 1:

Figure 2: