Mathematics 433 Fall 2007 Problems to Turn in: 6

November 2, 2007

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Name

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. **Only write on one side of each page.**

"*By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and, in effect, increases the mental power of the race.*" – Alfred North Whitehead

Problems

1. Prove the *Correspondence Theorem.*

   Theorem 1 *Let $\phi : G \to G'$ be an onto homomorphism between groups G and G' and with $\ker(\phi) = N$.Show the set of subgroups of G', $S = \{H' : H' \leq G'\}$ is in one-to-one correspondence with the set $T = \{H : H \leq G \text{ and } N \subset H\}$ of all subgroups of G that contain $N = \ker(\phi)$. I suggest you use the map $\lambda : S \to T$ where λ takes the subgroup H of G to the subgroup $\phi(H)$ of G'. That is, $\lambda(H) = \phi(H)$. Also prove that if H is a normal subgroup of G then $\lambda(H)$ is a normal subgroup of G'.*

   [It might be useful to explicitly work out the correspondence above in the special case when G is a cyclic group of order 12 generated by x, G' is a cyclic group of order 6 generated by y and ϕ is the map given by $\phi(x^i) = y^i$.

2. Do both of the following.
   (a) Prove the cartesian product of two infinite cyclic groups is not infinite cyclic.
   (b) Prove the center of the cartesian product of two groups is the cartesian product of their centers.

3. Do both of the following.
   (a) Prove every integer a is congruent to the sum of its digits modulo 9.
   (b) Prove the associative and commutative laws for multiplication in $\mathbf{Z}/n\mathbf{Z}$.

4. Prove the subset $G \times \{e'\}$ of the product group $G \times G'$ is a normal subgroup isomorphic to G. Also prove that
$$\frac{G \times G'}{G \times \{e'\}} \approx G'.$$

5. Let G be a finite group whose order is the product of two integers: $n = ab$. Let H, K be subgroups of G of orders a, b, respectively. Assume that $H \cap K = \{e\}$. Prove that $HK = G$. Is G isomorphic to $H \times K$?

6. Let H be a subgroup of a group G, and let $\phi : G \to H$ be a homomorphism whose restriction to H, $\phi|_H$, is the identity map. Let $N = \ker(\phi)$.
   (a) Prove that if G is abelian then it is isomorphic to the product group $H \times K$.
   (b) Without the assumption that G is abelian, find a bijective map $\psi : G \to H \times N$ and show by an example that G need not be isomorphic to the product group.