Math 181   Fall 2007   Quiz 07

November 13, 2007

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Name

"*Education: That which discloses to the wise and disguises from the foolish their lack of understanding.*"
-Ambrose Bierce, writer (1842-1914)

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1. (Section 8.5 #21) Does the following series converge or diverge? Give reasons for your answer.

$$\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$$

Answer: Using the ratio test we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{(n+1)!}{(2(n+1)+1)!}}{\frac{n!}{(2n+1)!}} \\
&= \lim_{n\to\infty} \frac{(n+1)!}{(2n+3)!} \cdot \frac{(2n+1)!}{n!} \\
&= \lim_{n\to\infty} \frac{(n+1)\, n!}{(2n+3)\,(2n+2)\,(2n+1)!} \cdot \frac{(2n+1)!}{n!} \\
&= \lim_{n\to\infty} \frac{(n+1)}{(2n+3)\,(2n+2)} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{\left(\frac{1}{n} + \frac{1}{n^2}\right)}{\left(2 + \frac{3}{n}\right)\left(2 + \frac{2}{n}\right)} \\
&= \frac{0}{4} = 0
\end{aligned}$$

Since this is the ratio test and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 < 1$ the series $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$ converges.

2. (Section 8.6 #24) Does the following series converge absolutely, converge conditionallly or diverge? Give reasons for your answer.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\sqrt[n]{10}\right)$$

Answer: Since $\lim_{n\to\infty} \left(\sqrt[n]{10}\right) = \lim_{n\to\infty} (10)^{1/n} = 10^0 = 1$ then we see that for very large values of n, $(-1)^{n+1} \left(\sqrt[n]{10}\right)$ oscillates between numbers that are very close to $+1$ and numbers that are very close to -1. Hence, $(-1)^{n+1} \left(\sqrt[n]{10}\right)$ is not limiting to any actual number and we must conclude that $\lim_{n\to\infty} (-1)^{n+1} \left(\sqrt[n]{10}\right)$ **does not exist.** Note that we do not have a nice notation that indicates why the limit does not exist.

3. (Section 8.7 #26) Find the radius and interval of convergence of the following series. For what values of x does the series converge absolutely? For what values of x does it converge conditionally?

$$\sum_{n=0}^{\infty} (-2)^n\,(n+1)\,(x-1)^n$$

Answer: We begin by running the Root Test on the absolute value series, $\sum_{n=0}^{\infty} 2^n (n+1) |x-1|^n$, associated with the given power series.

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{2^n (n+1) |x-1|^n} &= \lim_{n\to\infty} \sqrt[n]{2^n} \cdot \sqrt[n]{n+1} \cdot \sqrt[n]{|x-1|^n} \\ &= 2(1)|x-1| \end{aligned}$$

This tells us the original series will converge absolutely for any number x satisfying

$$\begin{aligned} 2|x-1| &< 1 \\ |x-1| &< \frac{1}{2} \\ -\frac{1}{2} &< x-1 < \frac{1}{2} \\ \frac{1}{2} &< x < \frac{3}{2} \end{aligned}$$

In addition we know that the original series diverges for any number x that satisfies either $x > \frac{3}{2}$ or $x < \frac{1}{2}$.

We now check the two remaining points ($x = \frac{1}{2}, \frac{3}{2}$) individually

(a) Substituting $x = \frac{1}{2}$ into the original series and using the fact that $(-2)^n \left(-\frac{1}{2}\right)^n = \left[(-2)\left(-\frac{1}{2}\right)\right]^n = 1^n = 1$, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} (-2)^n (n+1) \left(\frac{1}{2} - 1\right)^n &= \sum_{n=0}^{\infty} (-2)^n (n+1) \left(-\frac{1}{2}\right)^n \\ &= \sum_{n=0}^{\infty} (n+1) \end{aligned}$$

which diverges by the N'th term test since $\lim_{n\to\infty} (n+1) = \infty$ and hence is not zero.

(b) Substituting $x = \frac{3}{2}$ into the original series and using $(-2)^n \left(\frac{1}{2}\right)^n = \left[(-2)\left(\frac{1}{2}\right)\right]^n = (-1)^n$ we have

$$\begin{aligned} \sum_{n=0}^{\infty} (-2)^n (n+1) \left(\frac{3}{2} - 1\right)^n &= \sum_{n=0}^{\infty} (-2)^n (n+1) \left(\frac{1}{2}\right)^n \\ & \sum_{n=0}^{\infty} (-1)^n (n+1) \end{aligned}$$

which also diverges by the N'th term test because, similar to the answer in number 2 above, $(-1)^n (n+1)$ swings back and forth between very large positive numbers and very large negative numbers and so is not limiting to any number at all, let alone limiting to the number 0.