Due March 9

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Name

Be sure to re-read the WRITING GUIDELINES rubric, since it defines how your project will be graded. In particular, you may discuss this project with others but **you may not collaborate on the written exposition of the solution.**

"*Without education we are in a horrible and deadly danger of taking educated people seriously.*" (G.K. Chesterton)

Signal Reconstruction – A Beginning (How does the music get off the CD?)

Basic Background

When we hear music the tympanic membrane in our ears (the eardrum) vibrates because of changes in air pressure due to pressure waves produced by movement of a similar membrane in our stereo speakers. These pressure waves are described mathematically as the sum of sine and cosine functions that have different amplitudes and frequencies. So, phrased in a simplified form, a continuous function that is the sum of all of the sine and cosine functions describing the tones of what we hear (this continuous function is called a "signal") is produced by your stereo system and is used to move the speaker membrane to produce the pressure waves we "hear". In order for this to happen several complex things happen. The one associated with this project involves the sampling of the continuous function that describes the music signal.

The outputs of this continuous function are recorded ("sampled") at a finite number of times and the samples digitized so they can be recorded on a glass master that is used to produce compact disks. [The making of a glass master is a highly technical process where the digital information from the source is transferred in "PIT" form onto a polished glass that is covered with a photo resistant coating. The term "PIT" refers to a series of microscopic imprints that are formed on the photosensitive layer that coats that glass. As a laser beam comes in contact with the coated glass, it translates the digital information to the coating on the glass in the form of pits.]$_1$ There is much interesting mathematics behind the fact that, if one samples the **continuous** function at evenly spaced times that occur at least twice as often as the maximum frequency of all the sine and cosine functions whose sum makes up the signal then one can use that finite (and hence easily digitized) collection of samples to completely reconstruct the **continuous** signal. The mathematics used in this process involves discrete mathematics (because of the discrete nature of the samples) that we don't quite have the proper background to understand (when we finish Chapter 8 we will be better prepared.) However, the general mathematical processes used in this discrete mathematics have a continuous analogy where one uses definite integrals in a fashion for which we are completely prepared. This project is about that continuous analog.

Project Background

Any music that has been sampled and recorded on a compact disk has only a finite number of frequencies. All of the extremely high and extremely low frequencies are stripped off of the signal by the sampling process. [This is why there are music aficionados who pay $30,000 for turntable based music systems. Those old style records contain many more of the very high and low frequencies and these aficionados say they can hear the difference. However there are a number of studies that show that **most** people cannot distinguish the difference. I do not know who funded those studies ;-).] That these frequencies are stripped out means that when we work to reconstruct the continuous signal, we know exactly what frequencies actually occur in the sum of sine and cosine functions that make it up. One fact that can be shown, but that we will not show in this project, is that if we are reconstructing the continuous signal over a known time interval $0 \le t \le T$, then there is a "fundamental frequency, $f_0 = \frac{1}{T}$ with the property that every sine and cosine function in the sum of functions that make up our signal has a frequency that is an integer multiple of f_0. Hence the continuous signal, which we denote by $p(t)$, we are going to reconstruct can be written in the form

$$p(t) = \sum_{k=1}^{N} \left[A_k \cos(2\pi k f_0 t) + B_k \sin(2\pi k f_0 t)\right].$$

where T, N and f_0 are known beforehand but the amplitudes $A_1, A_2, \cdots, A_N$ and $B_1, B_2, \cdots, B_N$ are not. So, all we need to know in order for us to determine which sine and cosine functions make up the continuous signal $p(t)$ (which we need to know in order to reconstruct it) are these amplitudes. Seeing how integrals are used in doing this is the purpose of this project.

The Project Itself

Our task can be presented in a simple fashion. Suppose that you have the graph (say on an oscilloscope) of a continuous function $p(t)$ that denotes a continuous music signal but that you need to know the sine and cosine functions whose sum equals $p(t)$ in order to convert that function into a form your stereo system can use to generate the actual sounds. To do this you have access to a tool that can determine the number $\int_0^T f(t)\, dt$ for any function that can be graphed on your oscilloscope.

The Mathematics of the Solutions

See the Student Submission on the web page for a model of how to write up the mathematics.

1. Compute the integrals $\int_0^T \cos(2\pi m f_0 t)\cos(2\pi n f_0 t)\, dt$, $\int_0^T \cos(2\pi m f_0 t)\sin(2\pi n f_0 t)\, dt$, and $\int_0^T \sin(2\pi m f_0 t)\sin(2\pi n f_0 t)\, dt$ where m and n are **different** integers. Use integration by parts at least once and show your work

   We will use integration by parts in the first integral and trigonometric identities in the other two.

   (a) Let $z = 2\pi f_0 t$ so that $dz = 2\pi f_0\, dt$. Then

   $$\int_0^T \cos(2\pi m f_0 t)\cos(2\pi n f_0 t)\, dt = \frac{T}{2\pi}\int_0^{2\pi} \cos(mz)\cos(nz)\, dz$$

   Now, setting $I = \int_0^T \cos(2\pi m f_0 t)\cos(2\pi n f_0 t)\, dt$ and using integration by parts with $u = \cos(mz)$ and $dv = \cos(nz)\, dz$ we see that $du = -m\sin(mz)\, dz$ and $v = \frac{1}{n}\sin(nz)$ so that

$$\begin{aligned}
I &= \frac{T}{2\pi}\int_0^{2\pi} \cos(mz)\cos(nz)\, dz \\
&= \frac{T}{2\pi}\left\{\left[\cos(mz)\frac{1}{n}\sin(nz)\right]_0^{2\pi} - \int_0^{2\pi}\frac{1}{n}\sin(nz)(-m)\sin(mz)\, dz\right\} \\
&= \frac{T}{2\pi}\left\{\left[\cos(mz)\frac{1}{n}\sin(nz)\right]_0^{2\pi} - \int_0^{2\pi}\frac{1}{n}\sin(nz)(-m)\sin(mz)\, dz\right\} \\
&= \frac{T}{2\pi}\left\{\frac{1}{n}\left[\cos(mz)\sin(nz)\right]_0^{2\pi} - \frac{-m}{n}\int_0^{2\pi}\sin(nz)\sin(mz)\, dz\right\} \\
&= \frac{T}{2\pi}\left\{\frac{1}{n}\left[\cos(2\pi m)\sin(2\pi z)\right] - \frac{1}{n}\left[\cos(0)\sin(0)\right] + \frac{m}{n}\int_0^{2\pi}\sin(nz)\sin(mz)\, dz\right\} \\
&= \frac{T}{2\pi}\left\{\frac{1}{n}\left[(1)(0)\right] - \frac{1}{n}\left[(1)(0)\right] + \frac{m}{n}\int_0^{2\pi}\sin(nz)\sin(mz)\, dz\right\} \\
&= \frac{T}{2\pi}\left(\frac{m}{n}\right)\int_0^{2\pi}\sin(nz)\sin(mz)\, dz
\end{aligned}$$

Now in this second integral we again use integration by parts with $u_1 = \sin(mz)$ and $dv_1 = \sin(nz)\ dz$ we see that $du_1 = m\cos(mz)$ and $v_1 = -\frac{1}{n}\cos(nz)\ dz$. Substituting this back into the previous equation we get

$$\begin{aligned}
I &= \frac{T}{2\pi}\left(\frac{m}{n}\right)\int_0^{2\pi} \sin(nz)\sin(mz)\ dz \\
&= \frac{T}{2\pi}\left(\frac{m}{n}\right)\left\{\left(\frac{-1}{n}\right)[\cos(nz)\sin(mz)]_0^{2\pi} - \int_0^{2\pi} \frac{-1}{n}\cos(nz)\,m\cos(mz)\ dz\right\} \\
&= \frac{T}{2\pi}\left(\frac{m}{n}\right)\left\{\frac{-1}{n}[\cos(2\pi n)\sin(2\pi n)] - \left(\frac{-1}{n}\right)[\cos(0)\sin(0)] + \int_0^{2\pi} \frac{1}{n}\cos(nz)\,m\cos(mz)\ dz\right\} \\
&= \frac{T}{2\pi}\left(\frac{m}{n}\right)\left\{\frac{-1}{n}[(1)(0)] - \left(\frac{-1}{n}\right)[(1)(0)] + \int_0^{2\pi} \frac{1}{n}\cos(nz)\,m\cos(mz)\ dz\right\} \\
&= \frac{T}{2\pi}\left(\frac{m^2}{n^2}\right)\int_0^{2\pi} \cos(nz)\cos(mz)\ dz \\
&= \left(\frac{m^2}{n^2}\right) I
\end{aligned}$$

This implies that $I\left(1 - \frac{m^2}{n^2}\right) = 0$ and since $m \neq n$ so that we can divide by $1 - \frac{m^2}{n^2} \neq 0$ we obtain $I = 0$.

(b) It is also possible to compute the above integral using trigonometric identies. Since $\cos(mx)\cos(nx) = \frac{1}{2}[\cos((m-n)x) + \cos((m+n)x)]$ we see that

$$\cos(2\pi m f_0 t)\cos(2\pi n f_0 t) = \begin{cases} \frac{1}{2}[\cos(2\pi f_0(m-n)t) + \cos((2\pi f_0(m+n)t))], & \text{if } m \neq n \\ \frac{1}{2}[1 + \cos(4\pi f_0 m t)], & \text{if } m = n \end{cases}$$

So, using substitutions $u = 2\pi f_0(m-n)t$ and $w = 2\pi f_0(m+n)t$, we get

$$\begin{aligned}
\int_0^T \cos(2\pi m f_0 t)\cos(2\pi n f_0 t)\ dt &= \begin{cases} \int_0^T \frac{1}{2}[\cos(2\pi f_0(m-n)t) + \cos((2\pi f_0(m+n)t))]\,dt, & \text{if } m \neq n \\ \int_0^T \frac{1}{2}[1 + \cos(4\pi f_0 m t)]\ dt, & \text{if } m = n \end{cases} \\
&= \begin{cases} \frac{1}{2}\left[\frac{\sin(2\pi f_0(m-n)t)}{2\pi f_0(m-n)} + \frac{\sin((2\pi f_0(m+n)t))}{2\pi f_0(m+n)}\right]_0^{T=1/f_0}, & \text{if } m \neq n \\ \frac{1}{2}\left[t + \frac{\sin(4\pi f_0 m t)}{4\pi f_0 m}\right]_0^{T=1/f_0}, & \text{if } m = n \end{cases} \\
&= \begin{cases} \frac{1}{2}\left[\frac{\sin(2\pi(m-n))}{2\pi f_0(m-n)} + \frac{\sin((2\pi(m+n)))}{2\pi f_0(m+n)}\right] - \frac{1}{2}[0-0], & \text{if } m \neq n \\ \frac{1}{2}\left[T + \frac{\sin(4\pi m)}{4\pi f_0 m}\right] - \frac{1}{2}\left[0 - \frac{\sin(0)}{4\pi f_0 m}\right], & \text{if } m = n \end{cases} \\
&= \begin{cases} 0, \text{ if } m \neq n \\ \frac{1}{2}T, \text{ if } m = n \end{cases}
\end{aligned}$$

(c) Similarly, since $\sin(mx)\sin(nx) = \frac{1}{2}[\cos((m-n)x) - \cos((m+n)x)]$ and $\sin(mx)\cos(nx) = \frac{1}{2}[\sin((m-n)x) + \sin((m+n)x)]$, we have

$$\begin{aligned}
\sin(2\pi m f_0 t)\sin(2\pi n f_0 t) &= \begin{cases} \frac{1}{2}[\cos(2\pi f_0(m-n)t) - \cos((2\pi f_0(m+n)t))], & \text{if } m \neq n \\ \frac{1}{2}[1 - \cos(4\pi f_0 m t)], & \text{if } m = n \end{cases} \\
\sin(2\pi m f_0 t)\cos(2\pi n f_0 t) &= \begin{cases} \frac{1}{2}[\sin(2\pi f_0(m-n)t) + \sin((2\pi f_0(m+n)t))], & \text{if } m \neq n \\ \frac{1}{2}[0 + \sin(4\pi f_0 m t)], & \text{if } m = n \end{cases}
\end{aligned}$$

Using the same substitutions as in part a. we get

$$\int_0^T \sin(2\pi m f_0 t)\sin(2\pi n f_0 t)\ dt = \begin{cases} \int_0^T \frac{1}{2}[\cos(2\pi f_0(m-n)t) - \cos((2\pi f_0(m+n)t))]\,dt, & \text{if } m \neq n \\ \int_0^T \frac{1}{2}[1 - \cos(4\pi f_0 m t)]\ dt, & \text{if } m = n \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left[\frac{\sin(2\pi f_0(m-n)t)}{2\pi f_0(m-n)} - \frac{\sin((2\pi f_0(m+n)t))}{2\pi f_0(m+n)}\right]_0^{T=1/f_0}, & \text{if } m \neq n \\ \frac{1}{2}\left[t - \frac{\sin(4\pi f_0 m t)}{4\pi f_0 m}\right]_0^{T=1/f_0}, & \text{if } m = n \end{cases}$$

$$= \begin{cases} 0, \text{ if } m \neq n \\ \frac{1}{2}T, \text{ if } m = n \end{cases}$$

and if $I = \int_0^T \sin(2\pi m f_0 t)\cos(2\pi n f_0 t)\ dt$ then

$$\begin{aligned}
I &= \begin{cases} \int_0^T \frac{1}{2}\left[\sin(2\pi f_0 (m-n)t) + \sin((2\pi f_0 (m+n)t))\right] dt, & \text{if } m \neq n \\ \int_0^T \frac{1}{2}\left[0 + \sin(4\pi f_0 m t)\right] dt, & \text{if } m = n \end{cases} \\
&= \begin{cases} \frac{1}{2}\left[\frac{-\cos(2\pi f_0(m-n)t)}{2\pi f_0(m-n)} + \frac{-\cos((2\pi f_0(m+n)t))}{2\pi f_0(m+n)}\right]_0^{T=1/f_0}, & \text{if } m \neq n \\ \frac{1}{2}\left[0 + \frac{-\cos(4\pi f_0 m t)}{4\pi f_0 m}\right]_0^{T=1/f_0}, & \text{if } m = n \end{cases} \\
&= \begin{cases} \frac{1}{2}\left[\frac{-\cos(2\pi(m-n))}{2\pi f_0(m-n)} + \frac{-\cos((2\pi(m+n)))}{2\pi f_0(m+n)}\right] - \frac{1}{2}\left[\frac{-\cos(0)}{2\pi f_0(m-n)} + \frac{-\cos(0)}{2\pi f_0(m+n)}\right], & \text{if } m \neq n \\ \frac{1}{2}\left[0 + \frac{-\cos(4\pi m)}{4\pi f_0 m}\right] - \frac{1}{2}\left[0 + \frac{-\cos(0)}{4\pi f_0 m}\right], & \text{if } m = n \end{cases} \\
&= \begin{cases} \frac{1}{2}\left[\frac{-1}{2\pi f_0(m-n)} + \frac{-1}{2\pi f_0(m+n)}\right] - \frac{1}{2}\left[\frac{-1}{2\pi f_0(m-n)} + \frac{-1}{2\pi f_0(m+n)}\right], & \text{if } m \neq n \\ \frac{1}{2}\left[0 + \frac{-1}{4\pi f_0 m}\right] - \frac{1}{2}\left[0 + \frac{-1}{4\pi f_0 m}\right], & \text{if } m = n \end{cases} \\
&= \begin{cases} 0, & \text{if } m \neq n \\ 0, & \text{if } m = n \end{cases}
\end{aligned}$$

2. Compute the integrals $\int_0^T \cos(2\pi m f_0 t)\cos(2\pi n f_0 t)\ dt$, $\int_0^T \cos(2\pi m f_0 t)\sin(2\pi n f_0 t)\ dt$, and $\int_0^T \sin(2\pi m f_0 t)\sin(2\pi n f_0 t)\ dt$ where m and n are **the same** integer. Show your work.Compute the integrals $\int_0^T \cos(2\pi m f_0 t)\cos(2\pi n f_0 t)\ dt$, $\int_0^T \cos(2\pi m f_0 t)\sin(2\pi n f_0 t)\ dt$, and $\int_0^T \sin(2\pi m f_0 t)\sin(2\pi n f_0 t)\ dt$ where m and n are **the same** integer. Show your work.

   We did this above when we kept track of the trigonometric identities both when $n \neq m$ and when $n = m$.

3. Use parts 1. and 2. to simplify the integrals of $p(t)\cos(2\pi m f_0 t)$ and $p(t)\sin(2\pi m f_0 t)$. If you do this correctly you should now have formulas that give you the values of the amplitudes A_k, $1 \leq k \leq N$ and B_k, $1 \leq k \leq N$ in terms of T, $\int_0^T p(t)\cos(2\pi m f_0 t)\ dt$, and $\int_0^T p(t)\sin(2\pi m f_0 t)\ dt$.

   We will use the fact that the integral of a sum of functions is the sum of the corresponding integrals which allows us to interchange integral signs and sigmas. Note that

$$\begin{aligned}
\int_0^T p(t)\cos(2\pi m f_0 t)\ dt &= \int_0^T \left[\sum_{k=1}^N \left[A_k \cos(2\pi k f_0 t) + B_k \sin(2\pi k f_0 t)\right]\right] \cos(2\pi m f_0 t)\ dt \\
&= \int_0^T \sum_{k=1}^N A_k \cos(2\pi k f_0 t)\cos(2\pi m f_0 t) + B_k \sin(2\pi k f_0 t)\cos(2\pi m f_0 t)\ dt \\
&= \sum_{k=1}^N \left[\int_0^T A_k \cos(2\pi k f_0 t)\cos(2\pi m f_0 t) + B_k \sin(2\pi k f_0 t)\cos(2\pi m f_0 t)\right] dt \\
&= \sum_{k=1}^N \left[A_k \int_0^T \cos(2\pi k f_0 t)\cos(2\pi m f_0 t)\ dt + B_k \int_0^T \sin(2\pi k f_0 t)\cos(2\pi m f_0 t)\right] dt \\
&= A_m \frac{T}{2} + 0 + 0 + 0 + \cdots + 0
\end{aligned}$$

Where the zeros in the last expression come from our work in part 1 and 2 which shows the only nonzero integral occurs when $k = m$.

Thus we can solve for the amplitude A_m obtaining

$$A_m = \frac{2}{T} \int_0^T p(t) \cos(2\pi m f_0 t)\ dt$$

In a similar fashiom we can solve for B_m by noting

$$\begin{aligned}
\int_0^T p(t) \sin(2\pi m f_0 t)\ dt &= \int_0^T \left[\sum_{k=1}^{N} \left[A_k \cos(2\pi k f_0 t) + B_k \sin(2\pi k f_0 t)\right] \right] \sin(2\pi m f_0 t)\ dt \\
&= \int_0^T \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t) \sin(2\pi m f_0 t) + B_k \sin(2\pi k f_0 t) \sin(2\pi m f_0 t)\ dt \\
&= \sum_{k=1}^{N} \left[\int_0^T A_k \cos(2\pi k f_0 t) \sin(2\pi m f_0 t) + B_k \sin(2\pi k f_0 t) \sin(2\pi m f_0 t) \right] dt \\
&= \sum_{k=1}^{N} \left[A_k \int_0^T \cos(2\pi k f_0 t) \sin(2\pi m f_0 t)\ dt + B_k \int_0^T \sin(2\pi k f_0 t) \sin(2\pi m f_0 t) \right] dt \\
&= 0 + 0 + \ldots + 0 + B_m \frac{T}{2}
\end{aligned}$$

Thus, solving for the amplitude B_m we have

$$B_m = \frac{2}{T} \int_0^T p(t) \sin(2\pi m f_0 t)\ dt$$

4. Since you have access to a tool that can evaluate the numbers represented by these last integrals, you can determine the amplitudes A_k and B_k and store them on the compact disk so the stereo tuner can use them to reconstruct the signal and send it to the speakers for you to hear. Illustrate this process by writing out the function $p(t)$ describing the signal that has $N = 3$, $T = 6$, $A_1 = 2$, $A_2 = 4$, $A_3 = 0$, $B_1 = 3$, $B_2 = 0$, and $B_3 = 6$.

   Since $p(t) = \sum_{k=1}^{N} \left[A_k \cos(2\pi k f_0 t) + B_k \sin(2\pi k f_0 t)\right]$ we make the various substitutions to obtain

$$\begin{aligned}
p(t) &= \sum_{k=1}^{N} \left[A_k \cos(2\pi k f_0 t) + B_k \sin(2\pi k f_0 t)\right] \\
&= \sum_{k=1}^{3} \left[A_k \cos\left(2\pi k \frac{1}{6} t\right) + B_k \sin\left(2\pi k \frac{1}{6} t\right) \right] \\
&= A_1 \cos\left(\frac{2\pi}{6} t\right) + B_1 \sin\left(\frac{2\pi}{6} t\right) \\
&\quad + A_2 \cos\left(\frac{4\pi}{6} t\right) + B_2 \sin\left(\frac{4\pi}{6} t\right) \\
&\quad + A_3 \cos\left(\frac{6\pi}{6} t\right) + B_3 \sin\left(\frac{6\pi}{6} t\right) \\
&= 2 \cos\left(\frac{2\pi}{6} t\right) + 3 \sin\left(\frac{2\pi}{6} t\right) \\
&\quad + 4 \cos\left(\frac{4\pi}{6} t\right) + (0) \sin\left(\frac{4\pi}{6} t\right) \\
&\quad + (0) \cos\left(\frac{6\pi}{6} t\right) + 6 \sin\left(\frac{6\pi}{6} t\right)
\end{aligned}$$

Sources

1. http://www.emimusic.ca/making_cd.asp

- http://www.st-andrews.ac.uk/~jcgl/Scots_Guide/iandm/part7/page3.html
- http://en.wikipedia.org/wiki/Nyquist%E2%80%93Shannon_sampling_theorem
- http://cnx.org/content/m10791/latest/