Mathematics 181 – B **Fall 2007**

Exam 2

October 9, 2007

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Name

Technology used: ______________________ Directions:

- Be sure to include in-line citations every time you use technology.
- Include a careful sketch of any graph obtained by technology in solving a problem.
- **Only write on one side of each page.**

Do all of the following problems

1. Evaluate any **three** (3) of the following.

(a) $\int \frac{1}{x^2} e^{1/x} \sec\left(2 + e^{1/x}\right) \tan\left(2 + e^{1/x}\right)\ dx$
Answer: $u = 2 + e^{1/x}$ so $du = -x^{-2} e^{1/x} dx$ so
$\int \frac{1}{x^2} e^{1/x} \sec\left(2 + e^{1/x}\right) \tan\left(2 + e^{1/x}\right)\ dx = -\int \sec(u)\tan(u)\,du = -\sec(u) + C = -\sec\left(2 + e^{1/x}\right) + C$

(b) $\int \frac{1}{x} \sin^2(\ln(x))\ dx$
Answer: $u = \ln(x)$ so $du = \frac{1}{x} dx$ and
$\int \frac{1}{x} \sin^2(\ln(x))\ dx = \int \sin^2(u)\,du = \frac{1}{2}\int (1 - \cos(2u))\,du = \frac{1}{2}\left[u - \frac{1}{2}\sin(2u)\right] + C$
$= \frac{1}{2}\ln(x) - \frac{1}{4}\sin(2\ln(x)) + C$

(c) $\int_{\ln(4)}^{\ln(9)} e^{x/2}\ dx$
Answer: $u = x/2$ so $\int_{\ln(4)}^{\ln(9)} e^{x/2}\ dx = 2e^{x/2}\big|_{2\ln(2)}^{2\ln(3)} = 2\left[e^{\frac{1}{2}\cdot 2\ln(3)} - e^{\frac{1}{2}\cdot 2\ln(2)}\right]$
$= 2\left[e^{\ln(3)} - e^{\ln(2)}\right] = 2[3 - 2] = 2$

(d) $\frac{d}{dx}\int_{e^{x^2}}^{2} \frac{1}{\sqrt{t}}\ dt$
Answer: $\frac{d}{dx}\int_{e^{x^2}}^{2} \frac{1}{\sqrt{t}}\ dt = -\frac{d}{dx}\int_{2}^{e^{x^2}} \frac{1}{\sqrt{t}} dt =$ by the first part of the FTC
$= -\frac{1}{\sqrt{e^{x^2}}} \cdot \frac{d}{dx}\left[e^{x^2}\right] = -e^{-\frac{1}{2}x^2} \cdot e^{x^2}(2x) = -2xe^{\frac{1}{2}x^2}$

2. The base of a solid is the region bounded by the graphs of $y = \sec(x)$, $y = 0$, $x = 0$ and $x = \pi/4$. The cross sections perpendicular to the x-axis are **semi**circles . Find the volume.
Answer: The diameter of the half circle at x is $\sec(x)$ so the radius is $\frac{\sec(x)}{2}$. Thus the volume satisfies
$V = \frac{1}{2}\pi \int_0^{\pi/4} \frac{\sec^2(x)}{4} dx = \frac{1}{8}\pi \int_0^{\pi/4} \sec^2(x)\,dx = \frac{1}{8}\pi \tan(x)\big|_0^{\pi/4} = \frac{1}{8}\pi[1 - 0] = \frac{\pi}{8}$.

3. A solid of revolution is formed when the region bounded by the curves $x = y^2$ and $x = 6 - y$ is rotated about the line $y = 4$. Use the method of cylindrical shells to find the volume.
Answer: The radius of the shell at level y is $(4 - y)$ and the height of that shell is $(6 - y - y^2)$ so the volume satisfies
$V = 2\pi \int_{-3}^{2} (4 - y)(6 - y - y^2)\ dy = 2\pi \int (24 - 10y - 3y^2 + y^3)\,dy$
$= 2\pi\left[\left(24y - 5y^2 - y^3 + \frac{1}{4}y^4\right)\right]_{-3}^{2} = \frac{375}{2}\pi$ which is about 589.0486

4. Find the length of the parametrized curve $x = \frac{t^3}{6} + \frac{1}{2t}$, $y = t$, from $t = 2$ to $t = 3$.

Answer: $\frac{dx}{dt} = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$ and $\frac{dy}{dt} = 1$ so $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(\frac{1}{2}x^2 - \frac{1}{2}x^{-2}\right)^2 + 1}$

$= \sqrt{\left(\frac{1}{2}x^2 + \frac{1}{2}x^{-2}\right)}$ so the length of the curve satisfies

5. $S = \int_2^3 \sqrt{\left(\frac{1}{2}t^2 + \frac{1}{2}t^{-2}\right)^2} dt = \int_2^3 \left|\frac{1}{2}t^2 + \frac{1}{2}t^{-2}\right| dt = \int_2^3 \left(\frac{1}{2}t^2 + \frac{1}{2}t^{-2}\right) dt = \frac{13}{4}$

6. Solve the separable differentiable equation

$$\sqrt{x}\frac{dy}{dx} = e^{y+\sqrt{x}}, \ x > 0.$$

Answer (Section B): Separating variables and making the substitution $u = \sqrt{x}$ gives

$$\begin{aligned}
\sqrt{x}\frac{dy}{dx} &= e^{y+\sqrt{x}} \\
\sqrt{x}\frac{dy}{dx} &= e^{y}e^{\sqrt{x}} \\
e^{-y}dy &= e^{\sqrt{x}}\frac{1}{\sqrt{x}}dx \\
\int e^{-y}dy &= \int e^{\sqrt{x}}\frac{1}{\sqrt{x}}dx \\
-e^{-y} &= 2e^{\sqrt{x}} + C
\end{aligned}$$

Extra credit was awarded for noticing that this cannot be solved for y since the left hand side is negative but the right side is positive for large x.

Answer (Section C): Mimicing the previous solution but with $\sqrt{x}\frac{dy}{dx} = e^{-y+\sqrt{x}}$ instead of $\sqrt{x}\frac{dy}{dx} = e^{y+\sqrt{x}}$

$$\begin{aligned}
\sqrt{x}\frac{dy}{dx} &= e^{\,-y+\sqrt{x}} \\
\sqrt{x}\frac{dy}{dx} &= e^{-y}e^{\sqrt{x}} \\
e^{y}dy &= e^{\sqrt{x}}\frac{1}{\sqrt{x}}dx \\
\int e^{y}dy &= \int e^{\sqrt{x}}\frac{1}{\sqrt{x}}dx \\
e^{y} &= 2e^{\sqrt{x}} + C \\
y &= \ln\left(2e^{\sqrt{x}} + C\right)
\end{aligned}$$

7. Do **one** of the following.

   (a) A wire in the shape of a semicircle of radius 7 has a density function $\delta(\theta) = 2\sin(\theta)\ \frac{\text{g}}{\text{cm}}$ that varies with the parameter angle θ. Use our three step process to set up a definite integral whose numerical value is the total mass (measured in grams) of the wire. Do not evaluate the integral. Use the parametric equations $x = 7\cos(\theta)$, $y = 7\sin(\theta)$, $0 \le \theta \le \pi$ where length is measured in centimeters.

   Answer: A typical piece of the wire has length Δs_k and density $\delta(\theta) = 2\sin(\theta)$ so its mass is about $\Delta m_k = 2\sin(\theta)\,\Delta s_k$. This gives the total mass as the integral $\int_0^{\pi} 2\sin(\theta)\,ds$ provided the integrand is integrable.

Since $ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta = \sqrt{[-7\sin(\theta)]^2 + [7\cos(\theta)]^2} d\theta = \sqrt{49} d\theta = 7 d\theta$ then we have the total mass given by

$M = \int_0^{\pi} 2\sin(\theta) \cdot 7 \cdot d\theta = 14 \int_0^{2\pi} \sin(\theta)\, d\theta = -14\cos(\theta)\,|_0^{\pi} = 28.$

(b) Empirical evidence indicates that the power dissipation in a hurricane is proportional to three things: the cube of the wind speed, the frictional drag from the surface area at the base of the hurricane and the surface air density. Assume that the wind velocity $V(r)$ depends only on the distance, r, from the center of the Hurricane and denote the outer radius of the hurricane by R, the surface drag coefficient by C_d, and the surface air density by ρ.Use this information and our three-step process to build a definite integral that represents the total power dissipation.

Answer: The power dissipation is proportional to $C_d \cdot 2\pi r \cdot \Delta r$ since the drag is essentially constant along the washer of radius r . Thus, the Riemann Sum approximating the power dissipation is $\sum_{k=1}^{n} K \cdot |V(r)|^3 \cdot \rho \cdot C_d 2\pi r \Delta r$ which leads to the integral

$\int_0^R K\,|V(r)|^3 \cdot \rho \cdot C_d 2\pi r\, dr.$