Conceptual Review

Sections 5.5, 5.6, (5.7), 6.1, 6.2, 6.3

Chapter 5 (after the Fundamental Theorem of Calculus)

Section 5.5: Indefinite Integrals, Substitution

- Using the Fundamental Theorem requires knowing the antiderivatives of the function to be integrated.
- The antiderivatives associated with reversing the known derivative rules are basic and collected in the back inside cover of the textbook.
- Antiderivatives for more sophisticated functions almost always are found by using the Method of Substitution to reverse a Chain Rule derivative.

The Technique:

- Make a substitution $u = g(x)$ and convert all of the 'x' parts of the indefinite integral into their 'u' equivalents.

What to Substitute

1. Substitute for the inside of the most complicated part of the integrand (works for all "straight-forward" problems)
2. If the above doesn't work, make a substitution that makes most of the integrand look like a known antiderivative formula. Then see if algebraic or trigonometric tweaks can convert the problem.
3. Look for a factor (a part being multiplied against all of the rest) of the integrand that is a known derivative $g'(x)$ and substitute for $g(x)$.

Section 5.6: Substitution and Area between Curves

- How to use substitution in **definite** integrals. Converting limits of integration from 'x' values into 'u' values.
- Area between curves is the first of many applications of the definite integral.
    1. The formula is developed naturally by building Riemann sums to approximate the given area and then using the requirement that the functions be continuous to deduce the area is the limit of those Riemann sums – that is, the area is given by a definite integral.
    2. Be sure to graph the functions first to see where the functions cross (if they do).
    3. Using horizontal rectangles in Riemann sums gives rise to formulas for integrating with respect to y.

Section 5.7**:** $\ln(x)$ **defined as a definite integral.**

- The "real" definition of the function $f(x) = \ln(x)$ is as an antiderivative of the only function of the form x^n that doesn't have an antiderivative of the form $\frac{1}{n+1}x^{n+1}+C$.Specifically, $\ln(x) = \int_1^x \frac{1}{t}\,dt$.

- This section uses this definition to show that all of the properties of $\ln(x)$ (and e^x) that you already know follow from this definition.

- If you go back to sections 1.4 and 1.5 of our text you will see that the explanation of the properties of e^x seems a bit contrived. The reason given for using the number e is that "it simplifies many of the calculations of calculus" (page 33). That should strike you as an explanation that doesn't really explain. Of course, the actual reason calculus oriented uses are simplified is that the "real" definition of e^x (found in section 5.7) is that it is the inverse function of $\ln(x)$ and the latter is actually **defined** to fill the $n = -1$ hole in the list of antiderivatives of x^n – a very calculus-oriented reason.

Chapter 6: Applications of the Definite Integral

This chapter is devoted to showing you that the definite integral is quite useful in many contexts. Specifically, there are many different quantities that can be estimated using Riemann sums and hence can be computed exactly using definite integrals.

Section 6.1**: Volumes by Slicing**

- If a solid is "nice" enough that all of the slices of the solid perpendicular to some axis can be represented by a function $A(x)$ (the cross-sectional area function), then the volume of the solid can be estimated to any degree of accuracy by a Riemann sum. Thus if A is continuous, the exact value is the limit of those Riemann sums – a definite integral.

- Cavalieri's Principle follows immediately from this definition: the volume of a solid is determined by the areas of its cross-sections and not their shape.

- Solids of revolution are introduced in this section because the area of their cross-sections is particularly easy to compute.

Section 6.2**: Volumes by Cylindrical Shells**

- Solids of revolution can also be approximated by Riemann sums of nested cylindrical shells. This gives rise to the definite integrals associated with the Method of Cylindrical Shells. Conceptually, this method has no more value than that of finding volumes of revolution by the Method of Slicing since they give equal answers. However, since each method requires finding an antiderivative in order to compute a volume, it is entirely possible (and quite common) that the antiderivative for one method is **much** harder to find than the antiderivative for the other. Which method is easier is highly dependent on the individual problem so there is no simple method for determining which one to use. In practice, one tries both and keeps the one that seems simpler.

Section 6.3: Lengths of Plane Curves

- This is our first example where our approximation, while a sum, is not quite a Riemann Sum. However the text, and I, have told you that more sophisticated techniques than what we know can be used to show that the limit of the given sum does exist and is equal to the definite integral given in this section.

- An important point to remember is that the tiny slanted line segments used to approximate the length of a **smooth** curve have length

  $$\Delta s_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

  and that the corresponding "differential" notation that occurs in the integrals is

  $$ds = \sqrt{dx^2 + dy^2}$$

- This gives the abbreviated formula for the length of a curve:

  $$L = \int_a^b ds$$

- Parametrized curves are also used in this section. Introduced in chapter 3 of the text (page 170), they were probably unfamiliar to most of you since the majority of calculus textbooks wait longer to bring them up. However, the basic idea is straightforward. A parametrized curve C with parametrization $x = f(t)$ and $y = g(t)$ is just a method of using one "parameter" to determine both the x and y coordinates of points on the curve. For example, if we use the standard parametrization for the circle centered at the origin of radius 1, we have $x(t) = \cos(t)$ and $y(t) = \sin(t)$, $0 \leq t \leq 2\pi$. Hence, one of the points on the circle (corresponding to the angle $t = \frac{\pi}{6}$) has coordinates $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ because $x\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $y\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. Other points can be found by using different values of t.