Mathematics 433-A Fall 2006

Exam 1

September 26, 2006

____
Name

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use a text or notes or technology. Include a careful sketch of any graph obtained by technology in solving a problem. **Only write on one side of each page.**

The Problems

1. $(10, 10$ points)

   (a) Use one of the principles of mathematical induction to prove if a, b, c are elements in a group G for which $b = cac^{-1}$, then $b^n = ca^nc^{-1}$ is true for all positive integers n.

      i. If $n = 1$ then $b^1 = ca^1c^{-1}$ is given.

      ii. Assume the formula holds for $n = k$.

      iii. Then $b^{k+1} = b^kb = \left(ca^kc^{-1}\right)(cac^{-1}) = ca^kac^{-1} = ca^{k+1}c^{-1}$
      So the formula holds for every positive integer by weak induction.

   (b) Prove that $b^n = ca^nc^{-1}$ is also true for all **negative** integers n.

      i. Note that $b^{-n} = (b^n)^{-1} = \left(ca^nc^{-1}\right)^{-1} = ca^{-n}c^{-1}$ so the formula holds for negative integers. (It holds for $n = 0$ trivially.)

   (c) Prove that if $\phi : G \to H$ is a group homomorphism then $\phi(a^{-1}) = (\phi(a))^{-1}$ for every $a \in G$

      i. $e' = \phi(e) = \phi(aa^{-1}) = \phi(a)\phi(a^{-1})$ and $e' = \phi(e) = \phi(a^{-1}a) = \phi(a^{-1})\phi(a)$ so $\phi(a^{-1})$ acts like the inverse of $\phi(a)$. Since inverses are unique $\phi(a^{-1}) = [\phi(a)]^{-1}$

2. $(10, 10$ points) Let G be a group and $\phi : G \to G$ be the map $\phi(x) = x^{-1}$.

   (a) Prove that ϕ is a bijection

      i. If $\phi(x) = \phi(y)$ then $x^{-1} = y^{-1}$ so $xx^{-1}y = xy^{-1}y$ which tells us $y = x$ and ϕ is one-to-one.

      ii. Let y be an element in the codomain, G. Then y^{-1} is in G (the domain) and $\phi(y^{-1}) = (y^{-1})^{-1} = y$ so is onto.

   (b) Prove that ϕ is an automorphism if and only if G is abelian.

      i. If G is abelian then $\phi(ab) = (ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1} = \phi(a)\phi(b)$ so ϕ respects the group operation.

      ii. If ϕ is an automorphism, then $a^{-1}b^{-1} = \phi(a)\phi(b) = \phi(ab) = (ab)^{-1} = b^{-1}a^{-1}$. Taking inverses of both ends of this equation yields $ba = ab$ so every pair of elements in G commute and G is abelian.

3. (15 points each) Do four (4) of the following problems.

   (a) Prove that every subgroup of a cyclic group is cyclic.

      i. In the text and in class notes.

   (b) Prove that a group in which every element except the identity has order 2 is abelian.

i. For any elements a, b in G, $ab = ab\,(ba)^2 = abbaba = ab^2aba = a^2ba = ba$.

(c) Find all automorphisms of the group $(Z, +)$ of integers under the operation of addition. [Recall that every subgroup of $(Z, +)$ has the form bZ.]

i. Let ϕ be an automorphism of the group $G = (Z, +)$. Then 1 maps to an integer, say m. And since ϕ is onto, there is an integer m that maps to 1.

ii. Thus, $1 = \phi(m) = m\phi(1) = nm$.

iii. Since both n, m are integers they are either both 1 or both -1.

iv. So $\phi(n) = n$ and $\psi(n) = -n$ are the only possible automorphisms.

v. It is now easy to check they are both automorphisms.

(d) (15 points) Let ϕ, ψ be two homomorphisms from a group G to another group G' and let $H \subset G$ be the subset of G given by $H = \{x \in G : \phi(x) = \psi(x)\}$ .Prove or disprove, H is a subgroup of G.

i. Let a, b be elements in H. Then $\phi(ab) = \phi(a)\phi(b) = \psi(a)\psi(b) = \psi(ab)$ so H is closed.

ii. Let a be an element of H. Then $\phi(a) = \psi(a)$ tells us $\phi(a^{-1}) = [\phi(a)]^{-1} = [\psi(a)]^{-1} = \psi(a^{-1})$ so H contains the inverse of each of its elements.

iii. Thus H is a subgroup of G.

(e) Let H be a subgroup of a group G. Prove that the relation defined by the rule $a\tilde{\ }b$ if and only if $b^{-1}a \in H$ is an equivalence relation on G.

i. Since $a^{-1}a = e \in H$, then $a\tilde{\ }a$

ii. If $a\tilde{\ }b$ then $b^{-1}a \in H$ and taking inverses, $(b^{-1}a)^{-1} = a^{-1}b \in H$ so $b\tilde{\ }a$

iii. If $a\tilde{\ }b$ and $b\tilde{\ }c$ then $b^{-1}a$, $c^{-1}b \in H$ so $(c^{-1}b)(b^{-1}a) = c^{-1}a \in H$ and $a\tilde{\ }c$.

(f) The orders of the elements in $U(20)$ and $U(24)$ are given in the tables below. Prove that these two groups are not isomorphic by proving that if $\phi : G \to H$ is an isomorphism, then the order of a must equal the order of $\phi(a)$, $|a| = |\phi(a)|$.

$U(20)$	1	3	7	9	11	13	17	19
Order	1	4	4	2	2	4	4	2

$U(24)$	1	5	7	11	13	17	19	23
Order	1	2	2	2	2	2	2	2

i. Let a be an element of G of order n and denote the order of $\phi(a)$ by m. Then $e = \phi(e) = \phi(a^n) = [\phi(a)]^n$ tells us that m must divide n.

ii. We know that ϕ^{-1} is also an isomorphism and the order of $\phi(a)$ is some integer, say m.Then $e = \phi^{-1}(e) = \phi^{-1}([\phi(a)]^m) = \phi^{-1}(\phi(a))^m = a^m$ which tells us that n must divide m. The only way two positive integers m and n can divide each other is if they are equal.

iii. Note that this proof tells us that either a and $\phi(a)$ both have infinite order or both have finite order.

Definitions you should know.

1. The **general linear** group of order n over the real numbers $GL(n, \mathbf{R})$.
2. The **center,** $Z(G)$, of a group G.
3. The **centralizer,** $C(a)$, of an element a in a group G.
4. A **normal** subgroup N of a group G.
5. A **homomorphism** from the group G to the group G'.