Mathematics 290-A Exam 4 Fall 2006

November 14, 2006

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Name

Technology used: ______________________

- **Only write on one side of each page.**
- **Use terminology correctly.**
- **Partial credit is awarded for correct approaches so justify your steps.**
- **When given a choice, be sure to specify which problem(s) you want graded.**

Do any two (2) of these computational problems

C.1. Show that $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ is an eigenvector for the matrix $\begin{bmatrix} 2 & -6 & 6 \\ 1 & 9 & -6 \\ -2 & 16 & -13 \end{bmatrix}$ and determine the corresponding eigenvalue.

(a) $\begin{bmatrix} 2 & -6 & 6 \\ 1 & 9 & -6 \\ -2 & 16 & -13 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -8 \end{bmatrix} = -4 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ so the eigenvalue is $\lambda = -4$.

C.2. Given the subspace V of $\mathbf{C}^4$ where $V = \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\rangle$, determine the dimension of the subspace $V^\perp$ by finding a basis for $V^\perp$.

(a)

$$\begin{aligned} V^\perp &= \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \left\langle \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle = 0 \text{ and } \left\langle \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\rangle = 0 \right\} \\ &= \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \ a + b = 0 \text{ and } a + 2b + 3c + 4d = 0 \right\} \\ &= \left\{ \begin{bmatrix} 3c + 4d \\ -3c - 4d \\ c \\ d \end{bmatrix} \right\} = \left\{ c \begin{bmatrix} 3 \\ -3 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 4 \\ -4 \\ 0 \\ 1 \end{bmatrix} : c, d \in \mathbf{C} \right\} \end{aligned}$$

so $\left\{ c \begin{bmatrix} 3 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $V^\perp$ and the latter has dimension 2.

C.3. The characteristic polynomial of $A = \begin{bmatrix} -2 & -6 & -6 \\ -3 & 2 & -2 \\ 3 & 2 & 6 \end{bmatrix}$ is $P_A(x) = -(x+2)(x-4)^2$. Find all eigenvalues and determine their algebraic and geometric multiplicities.

(a) $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ so $A - (-2) I_3 = \begin{bmatrix} 0 & -6 & -6 \\ -3 & 4 & -2 \\ 3 & 2 & 8 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so

$E_A(-2) = \left\langle \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\rangle$

$A - 4I_3 = \begin{bmatrix} -6 & -6 & -6 \\ -3 & -2 & -2 \\ 3 & 2 & 2 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so $E_A(4) = \left\langle \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\rangle$

Thus, $\lambda = -2$ has algebraic multiplicity 1 and geometric multiplicity 1 and $\lambda = 4$ has algebraic multiplicity 2 and geometric multiplicity 1.

Do any two (2) of these problems from the text, homework, or class.

You may NOT just cite a theorem or result in the text. You must prove these results.

M.1. Prove Theorem RMRT, Rank of a Matrix is the Rank of the Transpose:

Suppose A is an $m \times n$ matrix. Then $r(A) = r(A^t)$.

(a) The proof is in the textbook.

M.2. From Project 11: Explain why the following 5×5 matrix that has a 3×3 zero submatrix is definitely singular (regardless of the 16 non-zeros marked by x's.)

$$A = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}$$

(a) We show $\det(A) = 0$ which implies A is singular. Note that expanding $\det = \begin{bmatrix} x & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix}$ along the first column gives $x \begin{vmatrix} 0 & x & x \\ 0 & x & x \\ 0 & x & x \end{vmatrix}$ which equals zero because of the column of all zeros.

Thus, expanding the determinant of A along the top row gives

$$\begin{aligned} \det(A) &= x \begin{vmatrix} x & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{vmatrix} - x \begin{vmatrix} x & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{vmatrix} + 0 - 0 \\ &= 0 - 0 \end{aligned}$$

M.3. Exercise T60 in subsection PD (Properties of Dimension): Suppose that W is a vector space with dimension 5, and U and V are subspaces of W, each of dimension 3. Prove that $U \cap V$ contains a non-zero vector.

(a) Proof in text.

Do two (2) of these problems you've not seen before.

T.1. Label the following statements as being true or false.

(a) The rank of a matrix is equal to the number of its nonzero columns. **False:** $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ has rank 1.

(b) The rank of a matrix is equal to the number of its nonzero rows. **False:** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ has rank 1.

(c) The $m \times n$ zero matrix is the only $m \times n$ matrix having rank 0. **True**

(d) Elementary row operations preserve rank. **True**

(e) An $n \times n$ matrix of rank n is invertible. **True**

(f) It is possible for a 3×5 matrix to have rank 4. **False:** a 3×5 can have at most 3 leading ones

(g) It is possible for a 5×3 matrix to have rank 4. **False:** a 5×3 can have at most 3 leading ones

T.2. Suppose that A is a 4×4 matrix with exactly two distinct eigenvalues, 5 and -9 and let $E_A(5)$ and $E_A(-9)$ be the corresponding eigenspaces, respectively.

(a) Write all possible characteristic polynomials of A that are consistent with $\dim(E_A(5)) = 3$

i. $1 \leq \gamma_A(\lambda) \leq \alpha_A(\lambda)$ and $\alpha_A(5) + \alpha_A(-9) = 4$ tells us that $P_A(x) = (x-5)^3(x+9)^1$

(b) Write all possible characteristic polynomials of A that are consistent with $\dim(E_A(-9)) = 2$

i. $1 \leq \gamma_A(\lambda) \leq \alpha_A(\lambda)$ and $\alpha_A(5) + \alpha_A(-9) = 4$ tells us that $P_A(x) = (x-5)^2(x+9)^2$ or $P_A(x) = (x-5)^1(x+9)^3$

T.3. A matrix A is idempotent if $A^2 = A$. Show that the only possible eigenvalues of an idempotent matrix are $\lambda = 0$ and $\lambda = 1$. Then give an example of a matrix that is idempotent and has both of these two values as eigenvalues.

(a) $A^2\vec{x} = A(A\vec{x}) = A(\lambda\vec{x}) = \lambda(A\vec{x}) = \lambda(\lambda\vec{x}) = \lambda^2\vec{x}$ and $A^2\ \vec{x} = A\vec{x} = \lambda\vec{x}$ tells us $\lambda^2\vec{x} = \lambda\vec{x}$ and since $\vec{x} \neq \vec{0}$, then $\lambda^2 = \lambda$ so $\lambda = 0$ or 1.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is idempotent and with eigenvalues 0 and 1.

T.4. An $n \times n$ matrix A is **nilpotent** if, for some positive integer k, $A^k = O$, where O denotes the $n \times n$ zero matrix. Prove that if A is nilpotent, then A is not invertible.

(a) Consider $0 = \det(O) = \det\left(A^k\right) = [\det(A)]^k$. Thus $\det(A) = 0$ and A is singular and hence not invertible.