Mathematics 122 A,D **Fall 2001** **Quiz 4**

Due September 28

Study Group Members **Name**

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. **Only write on one side of each page.**

"A life spent making mistakes is not only more honorable, but more useful than a life spent doing nothing."
– George Bernard Shaw

Problems

1. The geometric (graphical) meaning behind Simpson's rule for approximating a definite integral $\int_a^b f(x)dx$ is based on splitting the interval $[a, b]$ into subintervals and using a parabola to approximate the graph of f over each subinterval. The purpose of this problem is to supply the details missing from the textbook and to show that this graphical interpretation matches the weighted average of the Midpoint and Trapezoid Rules that we inferred in our class exercise.

   First we build some useful notation. Let n be the number of subintervals and let Δx be the length of each subinterval. We will use the endpoints and midpoints of each subinterval so denote these, in order, by $x_0, x_1, x_2, \ldots, x_{2n}$. With this notation, x_0 is the left endpoint of the first subinterval, x_1 is the midpoint of the first subinterval, and x_2 is the right endpoint of the first subinterval. Similarly, x_{2i-2}, x_{2i-1}, x_{2i} are the left endpoint, midpoint and right endpoint, respectively, of the i'th subinterval

   Now, use the following geometrically motivated steps to derive the Simpson's Rule formula.

   (a) Consider the parabola with equation $g(x) = ax^2 + bx + c$ that passes through the three points $(-h, z_0)$, $(0, z_1)$, (h, z_2) where h represents the positive constant $\frac{\Delta x}{2}$. Show that

   $$\begin{aligned} a &= \frac{z_0 - 2z_1 + z_2}{2h^2} \\ b &= \frac{z_2 - z_0}{2h} \\ c &= z_1 \end{aligned}$$

   (b) Use the Second Fundamental Theorem of Calculus to show that

   $$\int_{-h}^{h} g(x)\,dx = \frac{h}{3}\left(z_0 + 4z_1 + z_2\right).$$

   (c) Use this result to determine the area bounded by the parabola that passes through the three points (x_{2i-2}, y_{2i-2}), (x_{2i-1}, y_{2i-1}), and (x_{2i}, y_{2i}) corresponding to the left endpoint, midpoint and right endpoint of the i'th subinterval. Express your answer in terms of $f(x_{2i-2}), f(x_{2i-1}), f(x_{2i})$ and Δx. [Hint: change coordinates so the new vertical axis passes through the point with original coordinates (x_{2i-1}, y_{2i-1}).]

   (d) Show that if we use the parabola of the type shown in step (c) for each subinterval, then the sum of the net areas bounded by these parabolas is

   $$S_n = \frac{1}{6}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})\right]\Delta x.$$

   (e) Finally, show the formula derived in the previous step is exactly the same as the weighted average of the Midpoint Rule and Trapezoid Rule we found in the class exercise.